1. **Solution 1:** The solutions are $(4, 1)$, $(5, 2)$, and $(7, 10)$, corresponding to

$$\begin{aligned} 4! + 1 &= 25 = 5^2 = (1 \cdot 4 + 1)^2 \\ 5! + 1 &= 121 = 11^2 = (2 \cdot 5 + 1)^2 \\ 7! + 1 &= 5041 = 71^2 = (10 \cdot 7 + 1)^2. \end{aligned}$$

For $n \geq 8$ we have

$$\begin{aligned} n! + 1 &> n! \\ &\geq n(n-1)(n-2)(n-3)(n-4)(n-5) \cdot 2 \\ &> n^2 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \\ &= 360n^2 \end{aligned}$$

because $2n(n-1) > n^2$, and $n - 2 \geq 6, n - 3 \geq 5$, etc. Continuing, we have

$$n! + 1 > 121n^2 = (11n)^2 > (10n + 1)^2 \geq (kn + 1)^2,$$

where we use the condition $k \leq 10$ (and k positive) in the last step. Since we have the strict inequality $n! + 1 > (kn + 1)^2$ for $n \geq 8$, we can rule out any solutions in that range. That leaves $0 \leq n \leq 8$, and we can check that $n! + 1$ is not a square for $n = 0, 1, 2, 3, 6$. Therefore, the only solutions are the ones given for $n = 4, 5, 7$.

Note: The three solutions above are in fact the only known solutions to $n! + 1 = m^2$ for any integer m. It is an unsolved problem whether any more solutions exist, called Brocard's problem.

Solution 2: As in Solution 1, we first verify that $(4, 1)$, $(5, 2)$, and $(7, 10)$ are solutions. Then we use Legendre's formula: for a prime p,

$$v_p(m!) = \sum_{j \geq 1} \left\lfloor \frac{m}{p^j} \right\rfloor,$$

where $v_p(x)$ denotes the highest power of p dividing x.

The cases $n = 0, 1, 2$ give no solutions by direct inspection. We henceforth assume assume $n \geq 3$. Expanding our assumed equality gives

$$n! + 1 = k^2 n^2 + 2kn + 1,$$

so $n! = kn(kn + 2)$. Dividing by n, we get $(n - 1)! = k(kn + 2)$. Finally, let $m = n - 1$. Then $m \geq 2$, and the equation becomes

$$m! = k(km + k + 2).$$

Since $m \mid m!$, we obtain the divisibility condition

$$m \mid k(k + 2).$$

We now work through some cases.

- Suppose first that k has an odd prime divisor p. Since $p \mid k$, we have

$$kn + 2 \equiv 2 \pmod{p}.$$

Thus $p \nmid kn + 2$, and therefore

$$v_p(m!) = v_p((n-1)!) = v_p(k).$$

- Otherwise, suppose $k = 4$. We have

$$m! = (n-1)! = 4(4n+2) = 8(2n+1),$$

so

$$v_2(m!) = 3.$$

- Finally, for $k = 8$, we have

$$m! = (n-1)! = 8(8n+2) = 16(4n+1),$$

so

$$v_2(m!) = 4.$$

Using these valuations together with $m \mid k(k+2)$, we get the following finite list.

k	valuation condition	possible $m = n-1$	possible n
1	$m \mid 3$	3	4
2	$m \mid 8$	$2, 4, 8$	$3, 5, 9$
3	$v_3(m!) = 1,\ m \mid 15$	$3, 5$	$4, 6$
4	$v_2(m!) = 3,\ m \mid 24$	4	5
5	$v_5(m!) = 1,\ m \mid 35$	$5, 7$	$6, 8$
6	$v_3(m!) = 1,\ m \mid 48$	$3, 4$	$4, 5$
7	$v_7(m!) = 1,\ m \mid 63$	$7, 9$	$8, 10$
8	$v_2(m!) = 4,\ m \mid 80$	none	none
9	$v_3(m!) = 2,\ m \mid 99$	none	none
10	$v_5(m!) = 1,\ m \mid 120$	$5, 6, 8$	$6, 7, 9$

It remains only to check whether the listed candidates satisfy

$$(n-1)! = k(kn+2).$$

We compute

k	candidates for n	solutions
1	4	4
2	$3, 5, 9$	5
3	$4, 6$	none
4	5	none
5	$6, 8$	none
6	$4, 5$	none
7	$8, 10$	none
8	none	none
9	none	none
10	$6, 7, 9$	7

Therefore the only possible pairs are

$$(n, k) = (4, 1), (5, 2), (7, 10),$$

as desired.

2. We claim that no matter where Brutus starts, James will catch Brutus in $N - 1$ moves exactly if both play optimally.

   We first describe a winning strategy for James. James can maintain the following invariant:

   > Invariant I_k: Before Brutus's kth turn, for each coordinate i if James's position in that coordinate is J and Brutus's coordinate is B, then either $J = B$, or $1 \leq B < J \leq N + 1 - k$, or $k \leq J < B \leq N$.

   The invariant is true when the game begins, because before Brutus's 1st move, $N + 1 - k = N$ so the invariant is equivalent to $J = B$, $J > B$, or $J < B$. Now assume the invariant I_k is true for some k. After Brutus moves, in each coordinate, if Brutus chooses to move in the same value as James then James can keep his coordinate the same, and $J = B$ will hold. Otherwise, the same inequality will hold as before, and James can move one closer to Brutus, closing him off in a smaller rectangle in that direction. In particular, if $1 \leq B < J \leq N + 1 - k$ held before, then $1 \leq B < J \leq N + 1 - (k + 1)$ after James moves, and similarly if $k \leq J < B \leq N$ held before, $(k + 1) \leq J < B \leq N$ after James moves.

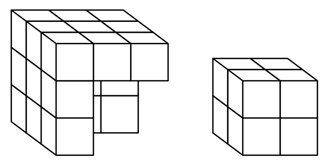
   By induction, the invariant must hold for all k. But once $k = N$, it becomes impossible for $1 \leq B < J \leq N + 1 - k = 1$ to hold, and similarly for $N = k \leq J < B \leq N$, so it must be that $J = B$ in both coordinates, prior to Brutus's Nth turn, which means that James will catch Brutus at worst in $N - 1$ turns.

   Brutus can in fact guarantee exactly $N - 1$ turns, since no matter where Brutus lies in the grid, in at least one coordinate, Brutus starts at least 1 away from James. In that coordinate, since Brutus plays first he can increase the distance to 2. The distance will continue to be 2 after Brutus's turn until

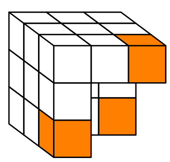
Brutus hits the edge of the grid, so it will take $N-1$ turns for James to reach Brutus since he has to cross the entire grid in that direction.

Therefore, $N-1$ is the number of moves for any point in the grid, so the expected value is $N-1$.

3. We split the $3\times3\times3$ cubical grid into two regions: the $2\times2\times2$ grid based at the front lower right corner (space for 8 cubes), and the shell that surrounds it to complete the $3\times3\times3$ grid (space for $3^3-2^3=19$ cubes).

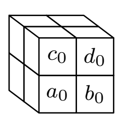


All of the colors of the cubes in the 19-cube grid are uniquely determined by the given views (for example, we know that whenever we see a white square in one of the side views, then the three cubes down that line must all be white; this shows that 16 of the cubes in our grid are white, and the remaining 3 cubes must be orange or else the views will not match). We illustrate this below.



Therefore, we need only determine the number of ways that we can color the eight remaining cubes in the $2\times2\times2$ grid so as to satisfy the conditions of the problem.

Suppose that we label the cubes on the front face of the $2\times2\times2$ grid as a_0, b_0, c_0, and d_0, as shown below, and we label the cubes behind them as a_1, b_1, c_1, and d_1, respectively.



If a_0 is colored orange, we write $a_0=1$, otherwise, we write $a_0=0$. We can write the constraints of each view as follows:

- Front View: $\max(a_0,a_1)=1$, $\max(c_0,c_1)=1$, $\max(d_0,d_1)=1$ (note that there is no restriction in this view on b_0 or b_1)
- Top View: $\max(a_0,c_0)=1$, $\max(a_1,c_1)=1$, $\max(b_1,d_1)=1$
- Left View: $\max(a_1,b_1)=1$, $\max(c_1,d_1)=1$, $\max(c_0,d_0)=1$

In particular, note that b_0 is not involved in any constraints, so its color can be chosen freely, and there are 2 choices for its color.

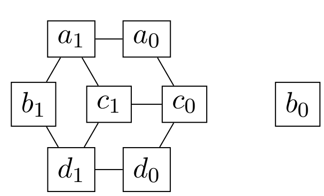
We now present two possible ways of completing the problem.

Method 1: *Casework.* We do casework on the values of (a_1, b_1, c_1, d_1), noting that from the left view, at least one of a_1 and b_1 is 1, and at least one of c_1 and d_1 is 1. Normally, we would have 3 ways to choose each of (a_1, b_1) and (c_1, d_1), namely $(1, 1)$, $(1, 0)$, or $(0, 1)$, which would yield $3^2 = 9$ cases. However, we also know from the top view, that at least one of a_1 and c_1 is 1, and at least one of b_1 and d_1 is 1, which eliminates 2 cases (namely $(a_1, b_1, c_1, d_1) = (1, 0, 1, 0)$ and $(0, 1, 0, 1)$). We can further combine two pairs of these cases, so we look at five cases.

- **Case 1:** $(a_1, b_1, c_1, d_1) = (1, x, 1, 1)$
  In this case, there are only two constraints that aren't immediately satisfied, namely $\max(a_0, c_0) = 1$ and $\max(c_0, d_0) = 1$. If $c_0 = 1$, then both remaining constraints are satisfied, and we can choose b_0, b_1, a_0, d_0 freely, so there are $2^4 = 16$ possibilities. If $c_0 = 1$, then we must have $a_0 = d_0 = 1$, so only b_0 and b_1 can be chosen freely, giving us $2^2 = 4$ possibilities. Thus there are $16 + 4 = 20$ possibilities in this case.
- **Case 2:** $(a_1, b_1, c_1, d_1) = (0, 1, 1, 1)$ or $(1, 1, 1, 0)$
  Note by symmetry that the two ordered pairs will yield the same number of possibilities. Therefore, we assume that $(a_1, b_1, c_1, d_1) = (1, 1, 1, 0)$. Since $d_1 = 0$, we know that $d_0 = 1$. From all of these values, only one of the constraints, namely $\max(a_0, c_0) = 1$, is not satisfied. There are $2^2 - 1 = 3$ ways to pick (a_0, c_0) such that $\max(a_0, c_0) = 1$, and only b_0 can be chosen freely (2 choices). Hence there are $3 \cdot 2 = 6$ possibilities in this case. We multiply this by 2 to account for the two ordered quadruples that are symmetric, getting a total of $6 \cdot 2 = 12$ possibilities.
- **Case 3:** $(a_1, b_1, c_1, d_1) = (1, x, 0, 1)$
  Since $c_1 = 0$, we know that $c_0 = 1$. These values satisfy all of the constraints of the problem, so b_1, a_0, b_0, and d_0 can be chosen freely, yielding $2^4 = 16$ possibilities.
- **Case 4:** $(a_1, b_1, c_1, d_1) = (0, 1, 1, 0)$
  Since $a_1 = 0$ and $d_1 = 0$, we know that $a_0 = 1$ and $d_0 = 1$. This is enough to satisfy all of the constraints, and so b_0 and c_0 can be chosen freely, giving $2^2 = 4$ possibilities.

Thus the answer is $20 + 12 + 16 + 4 = \boxed{52}$.

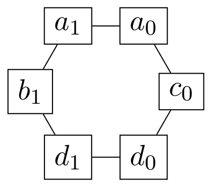
Method 2: *Graphs.* We can represent all of the above conditions as a graph with 8 vertices (representing the possible cubes), where two vertices are connected if and only if at least one of their corresponding cubes must be colored orange.



Note that since b_0 has no edges emanating from it, it can be colored orange or not colored orange, so it has 2 choices.

Now we do casework based on the color of c_1.

- If $c_1 = 0$, it forces $c_0 = a_1 = d_1 = 1$. But then each edge will be connected to at least one orange cube, so we can color the remaining three cubes however we want, and there are $2^3 = 8$ such colorings.
- If $c_1 = 1$, then the three edges coming out of c_1 all touch an orange vertex, so we can ignore those three edges and focus on the outer hexagonal grid:

  We wish to find the number of ways to color the vertices of a hexagon such that at least one vertex is colored on each edge. This can be computed in multiple ways—doing casework on the number of vertices colored, we find that there is 1 way if all six vertices are colored, 6 ways if five vertices are colored, $\binom{6}{2} - 6 = 9$ ways if four vertices are colored (place the two colored vertices, but subtract off arrangements where the two uncolored vertices are adjacent), and 2 ways if 3 vertices are colored. This gives a total of $1 + 6 + 9 + 2 = 18$ ways. Alternatively, one can show that the number of ways to color the vertices of a regular n-gon such that at least one vertex on each edge is colored is $F_{n+1} + F_{n-1}$, where n is the nth Fibonacci number (and $F_{n+1} + F_{n-1}$ is also the nth Lucas number L_n).

So the answer is

$$2 \cdot (2^3 + 18) = \boxed{52}.$$

4. **Solution 1:** The proposition is false. One counterexample is the sequence of all integers that do not have any 2s in their base-3 expansion (written in increasing order). The first few such numbers are:

   $$a_0 = 0, a_1 = 1, a_2 = 3 = 10_3, a_4 = 4 = 11_3, 9 = 100_3, a_5 = 10 = 101_3, \ldots.$$

   Then any number k can be decomposed digit-wise in base-3 by writing each two as a sum of two ones. For example, $70 = 2121_3 = 1111_3 + 1010_3 = 40 + 30$.

   Since the (2^n)th number in the sequence is $3^n = 100\cdots00_3$, there cannot exist a K such that $3^n \leq K2^n$ for all n, as $\left(\frac{3^n}{2^n}\right)$ grows arbitrarily large.

   Solution 2: Alternatively, we can consider the sequence of all nonnegative integers that can be written as the sum of two perfect squares. The first few such numbers are:

   $$a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 5, a_6 = 8, \ldots.$$

   It is well known that every positive integer can be written as the sum of four perfect squares. Hence a_i has the sum-of-two property.

On the other hand, it is also known that if $f(x)$ is the number of nonnegative integers n less than or equal to x such that n can be written as the sum of two squares, then $f(x) = O(x/\sqrt{\log x})$. In particular, if $f(x) \le Cx/\sqrt{\log x}$, we find that $f(a_i) = i+1$, so $i+1 \le Ca_i/\sqrt{\log a_i}$. Hence $a_i \ge (i+1)\sqrt{\log a_i}/C \ge (i+1)\sqrt{i}/C$. Since this is greater than a linear function times a square root, it will eventually be larger than any linear function, so no such constant K can exist.

Note: Similar to the first solution, we can also use ”the sequence of all integers which only use the digits $0-5$ in base 10” $(0, 1, 2, 3, 4, 5, 10, 11, 12,$ etc.). Another idea is $a_0 = 0$, $a_1 = 1$, and all numbers of the form p or $p+1$, where p is prime; this sequence has the sum-of-two property if and only if the Goldbach conjecture is true.

5. There are two optimal orderings:

$$(x_{n+1}, x_n, x_{n+2}, x_{n-1}, x_{n+3}, x_{n-2}, \dots, x_{2n}, x_1)$$

and

$$(x_n, x_{n+1}, x_{n+2}, x_{n-1}, x_{n+3}, x_{n-2}, \dots, x_{2n}, x_1).$$

Equivalently, let Fiona’s ordering be $(a_1, a_2, \dots, a_{2n})$. Then the optimal ordering satisfies

$$a_{2n} < a_{2n-2} < a_{2n-4} < \cdots < \begin{matrix} a_1 \\ a_2 \end{matrix} < a_3 < a_5 < \cdots < a_{2n-1},$$

where the placement of a_1 and a_2 means that a_1 and a_2 can freely switch places within the overall inequality chain, and this is what we prove.

In plain English, the largest n logs are placed in the odd-number steps (directly away from the stump) in increasing order and the smallest n logs are placed in the even-number steps (perpendicular) in decreasing order, and we may optionally swap the first two logs. We provide two solutions.

Solution 1: Given any positive integer k and an ordering of $2k$ logs $(a_1, \dots, a_{2k})$, let

$$f_k(a_1, a_2, \dots, a_{2k})$$

(which we shorten to f_k for convenience) denote Fiona’s distance from the stump after placing logs of lengths $a_1, \dots, a_{2k}$ by the process described in the problem statement. Taking also $f_0() = 0$, we have for all $k \ge 1$

$$f_k = \sqrt{(f_{k-1} + a_{2k-1})^2 + a_{2k}^2}.$$

We give a solution in three parts: first, we show that the odd-indexed log lengths must be in ascending order, then that the even-indexed log lengths must be in descending order, and finally that all the odd-indexed logs are longer than all the even-indexed logs (except swapping a_1 and a_2).

Call a permutation $(a_1, \dots, a_{2n})$ of $(x_1, \dots, x_{2n})$ optimal if $f_n(a_1, \dots, a_{2n})$ is maximal among all permutations.

We start with a lemma showing that f_n is increasing in each argument.

Lemma 1. *The function f_n is increasing in each argument.*

Proof. We induct on n. The base case is $n = 1$ and clearly $\sqrt{a_1^2 + a_2^2}$ increases if a_1 or a_2 increases. Assume that f_k is increasing in its arguments, and consider

$$f_{k+1}(a_1, \ldots, a_{2k+2}) = \sqrt{(f_k(a_1, \ldots, a_{2k}) + a_{2k+1})^2 + a_{2k+2}^2}.$$

If we increase a_{2k+2} or a_{2k+1} then clearly f_{k+1} increases. If we increase any of $a_1, \ldots, a_k$ then f_k increases by the inductive hypothesis, and f_{k+1} therefore increases by another application of the base case. □

Lemma 2. *If $f_k(a_1, \ldots, a_{2k}) > f_k(a'_1, \ldots, a'_{2k})$ then*

$$f_m(a_1, \ldots, a_{2k}, a_{2k+1}, \ldots, a_{2m}) > f_m(a'_1, \ldots, a'_{2k}, a_{2k+1}, \ldots, a_{2m})$$

for any $m \geq k$.

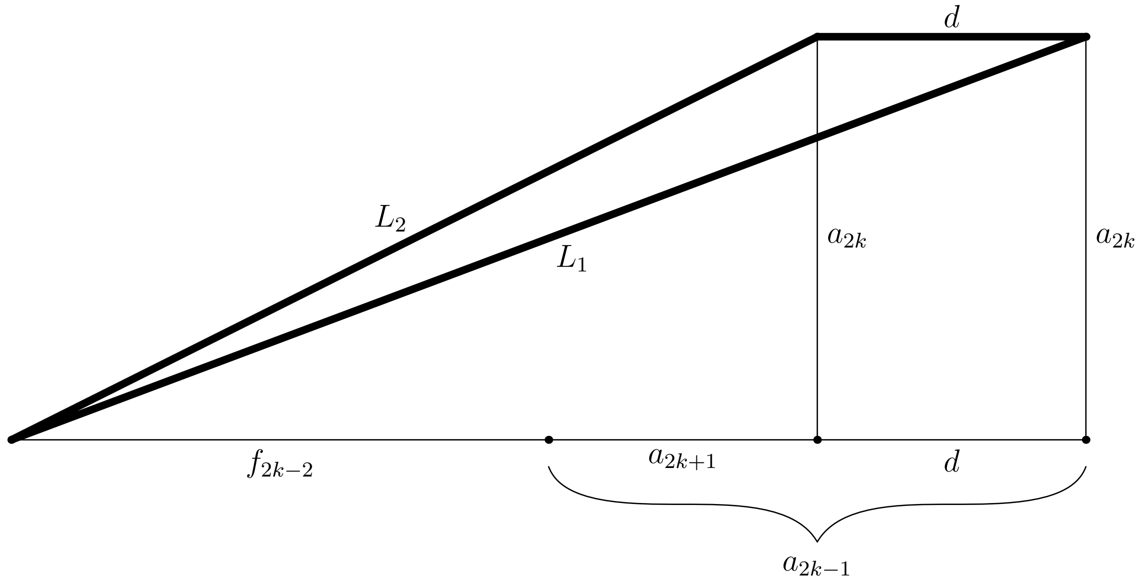
Proof. Let $j = m-k$. We induct on j. The base case $j = 0$ is assumed. The functions $f_j \mapsto f_j + a_{2j+1}$ and $g_j \mapsto \sqrt{g_j^2 + a_{2j+2}^2}$ are increasing in arguments f_j and g_j respectively, so their composition $f_j \mapsto f_{j+1}$ is increasing in f_j, completing the inductive step. □

Lemma 3. *The odd-indexed logs must be in increasing order. That is, let $1 \leq i < j \leq n$. Then in any optimal permutation $a_{2i-1} < a_{2j-1}$.*

Proof. Suppose for the sake of contradiction that, in an optimal permutation, there exist i and j such that $i < j$ but $a_{2i-1} > a_{2j-1}$. Then there is some k for which $a_{2k-1} > a_{2k+1}$. We compare this optimal permutation to the permutation formed by switching a_{2k-1} and a_{2k+1}; we have

$$\begin{aligned}
f_{k+1}(a_1, \ldots, a_{2k-1}, a_{2k}, a_{2k+1}, a_{2k+2}) &\geq f_{k+1}(a_1, \ldots, a_{2k+1}, a_{2k}, a_{2k-1}, a_{2k+2}) \\
\sqrt{\left(\sqrt{(f_{2k-2}+a_{2k-1})^2+a_{2k}^2}+a_{2k+1}\right)^2+a_{2k+2}^2} &\geq \sqrt{\left(\sqrt{(f_{2k-2}+a_{2k+1})^2+a_{2k}^2}+a_{2k-1}\right)^2+a_{2k+2}^2} \\
\sqrt{(f_{2k-2} + a_{2k-1})^2 + a_{2k}^2} + a_{2k+1} &\geq \sqrt{(f_{2k-2} + a_{2k+1})^2 + a_{2k}^2} + a_{2k-1} \\
\sqrt{(f_{2k-2}+a_{2k-1})^2+a_{2k}^2} - \sqrt{(f_{2k-2}+a_{2k+1})^2+a_{2k}^2} &\geq a_{2k-1} - a_{2k+1}.
\end{aligned}$$

Let $L_1 = \sqrt{(f_{2k-2} + a_{2k-1})^2 + a_{2k}^2}$, $L_2 = \sqrt{(f_{2k-2} + a_{2k+1})^2 + a_{2k}^2}$, and $d = a_{2k-1} - a_{2k+1}$. However, L_1, L_2, and d form the sides of a triangle as shown, so $L_1 - L_2 < d$, contradicting the above inequality and showing that swapping a_{2k-1} and a_{2k+1} puts Fiona further from the stump after $k + 1$ steps. By Lemma 2, Fiona will be further from the stump after n steps after the swap, so our permutation was not optimal. □



Lemma 4. *For each $2 \leq k \leq n$, we have*

$$f_k^2 = \left(\sum_{i=1}^{2k} a_i^2\right) + g_k(a_1, a_2, \ldots, a_{2k-1}),$$

where g_k is a positive function increasing in each of its arguments.

Proof. We use induction on k. We use $k = 1$ as the base case, with $f_2^2 = a_1^2 + a_2^2$ and $g_1 = 0$. We will add a strictly positive amount to g in each subsequent step so that $g_k > 0$ for $k \geq 2$, so this is a valid base case. Then, using the statement of the Lemma expanded to $k = 1$ and $g_1 = 0$ as the inductive hypothesis, we have

$$\begin{aligned}
f_{k+1}^2 &= (f_k + a_{2k+1})^2 + a_{2k+2}^2 \\
&= f_k^2 + a_{2k+1}^2 + a_{2k+2}^2 + 2f_k a_{2k+1} \\
&= \left(\sum_{i=1}^{2k+2} a_i^2\right) + g_k(a_1, \ldots, a_{2k-1}) + 2f_k a_{2k+1} \\
&= \left(\sum_{i=1}^{2k+2} a_i^2\right) + g_{k+1}(a_1, \ldots, a_{2k+1})
\end{aligned}$$

where

$$g_{k+1}(a_1, \ldots, a_{2k+1}) := g_k(a_1, \ldots, a_{2k-1}) + 2f_k a_{2k+1}.$$

Clearly f_k and a_{2k+1} are positive, so g_{k+1} is positive since g_k is at least nonnegative. Moreover, increasing any of $a_1, \ldots, a_{2k}$ increases f_k and subsequently g_{k+1}, while increasing a_{2k+1} increases the term $2f_k a_{2k+1}$, so g_{k+1} is increasing in all its arguments. □

Lemma 5. *The even-indexed logs must be in decreasing order. That is, let* $1 \le i < j \le n$*. Then in any optimal permutation* $a_{2i} > a_{2j}$.

Proof. We use strong induction on n. The base case is $n = 1$ and there is only one even-indexed log. Suppose that, for each $1 \le k < n$, all optimal permutations of the first $2k$ logs have even-indexed values in descending order. Now suppose that, for $2(k+1)$ logs, the smallest log is not the last one placed. To be precise, suppose $a_{2k+2} > a_{2j}$ for some $1 \le j \le k$. Consider swapping a_{2k+2} and a_{2j}; by the optimality assumption we have

$$f_{2k+2}(a_1, \ldots, a_{2j}, \ldots, a_{2k+2}) \ge f_{2k+2}(a_1, \ldots, a_{2k+2}, \ldots, a_{2j})$$
$$\left(\sum_{i=1}^{2k+2} a_i^2\right) + g_{2k+2}(a_1, \ldots, a_{2j}, \ldots, a_{2k+1}) \ge \left(\sum_{i=1}^{2k+2} a_i^2\right) + g_{2k+2}(a_1, \ldots, a_{2k+2}, \ldots, a_{2k+1})$$
$$g_{2k+2}(a_1, \ldots, a_{2j}, \ldots, a_{2k+1}) \ge g_{2k+2}(a_1, \ldots, a_{2k+2}, \ldots, a_{2k+1}),$$

but then, since $a_{2k+2} > a_{2j}$, g_{2k+2} is not increasing in each argument, contradicting Lemma 4. So the smallest log must go at the end, and by the inductive hypothesis all even-indexed logs must be in decreasing order. □

Lemma 6. *Let* $1 \le i, j \le n$ *but* $(i, j) \ne (1, 1)$*. In any optimal permutation* $a_{2i} < a_{2j-1}$.

Proof. Suppose that there exists an optimal permutation with $a_{2i} > a_{2j-1}$. We have two (not necessarily mutually exclusive) cases.

- $i \ne 1$. Then $a_2 > a_{2i} > a_{2j-1}$ by Lemma 5. Swapping a_1 and a_2 does not change the value of f_n, so there is another optimal permutation $(a'_1, \ldots, a'_{2n})$ with $a'_1 = a_2, a'_2 = a_1$, and $a'_k = a_k$ otherwise. Then $a'_1 > a'_{2j-1}$ which contradicts Lemma 3.
- $j \ne 1$. Then $a_{2i} > a_{2j-1} > a_1$ by Lemma 3. A similar swapping argument yields $a'_{2i} > a'_2$ contradicting Lemma 5. □

Putting Lemmas 3, 5, 6 together shows that, for any optimal permutation $(a_1, \ldots, a_{2k})$, we have

$$a_{2n} < a_{2n-2} < a_{2n-4} < \ldots a_2, a_1 < a_3 < a_5 < \cdots < a_{2n-1},$$

and the only permutations satisfying this chain of inequalities are the two as claimed.

Moreover, since there are finitely many possible permutations, an optimal permutation exists, and swapping a_1 and a_2 does not change the final distance, so both of these permutations are actually optimal.

Solution 2: In Solution 1, we replace the proof of Lemma 5 with the following (eliminating the need for Lemmas 1 and 4):

We use a direct swapping argument to show that the even-indexed logs must be in decreasing order. In an optimal permutation, suppose that $a_{2k} < a_{2k+2}$. Define $g_k = f_{k-1} + a_{2k-1}$ for any $k \geq 1$. By the optimality assumption and Lemma 2 we have

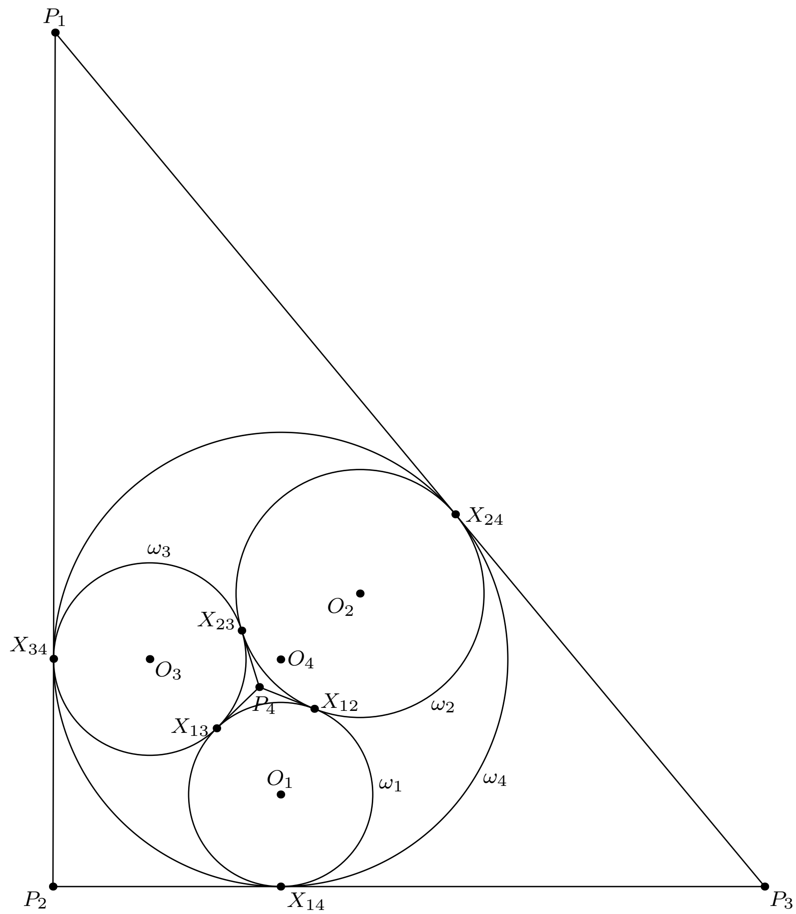
$$\left(\sqrt{g_k^2 + a_{2k}^2} + a_{2k+1}\right)^2 + a_{2k+2}^2 \geq \left(\sqrt{g_k^2 + a_{2k+2}^2} + a_{2k+1}\right)^2 + a_{2k}^2$$

Hence,

$$\begin{aligned}
&a_{2k+2}^2 - a_{2k}^2 \\
&\geq \left(\sqrt{g_k^2 + a_{2k+2}^2} + a_{2k+1}\right)^2 - \left(\sqrt{g_k^2 + a_{2k}^2} + a_{2k+1}\right)^2 \\
&= \left(2a_{2k+1} + \sqrt{g_k^2 + a_{2k+2}^2} + \sqrt{g_k^2 + a_{2k}^2}\right)\left(\sqrt{g_k^2 + a_{2k+2}^2} - \sqrt{g_k^2 + a_{2k}^2}\right) \\
&> \left(\sqrt{g_k^2 + a_{2k+2}^2} + \sqrt{g_k^2 + a_{2k}^2}\right)\left(\sqrt{g_k^2 + a_{2k+2}^2} - \sqrt{g_k^2 + a_{2k}^2}\right) \\
&= a_{2k+2}^2 - a_{2k}^2
\end{aligned}$$

where the crucial $>$ comes from $a_{2k+1} > 0$. This is a contradiction so the Lemma is proved.

6. **Solution 1:** One possible configuration is shown below.



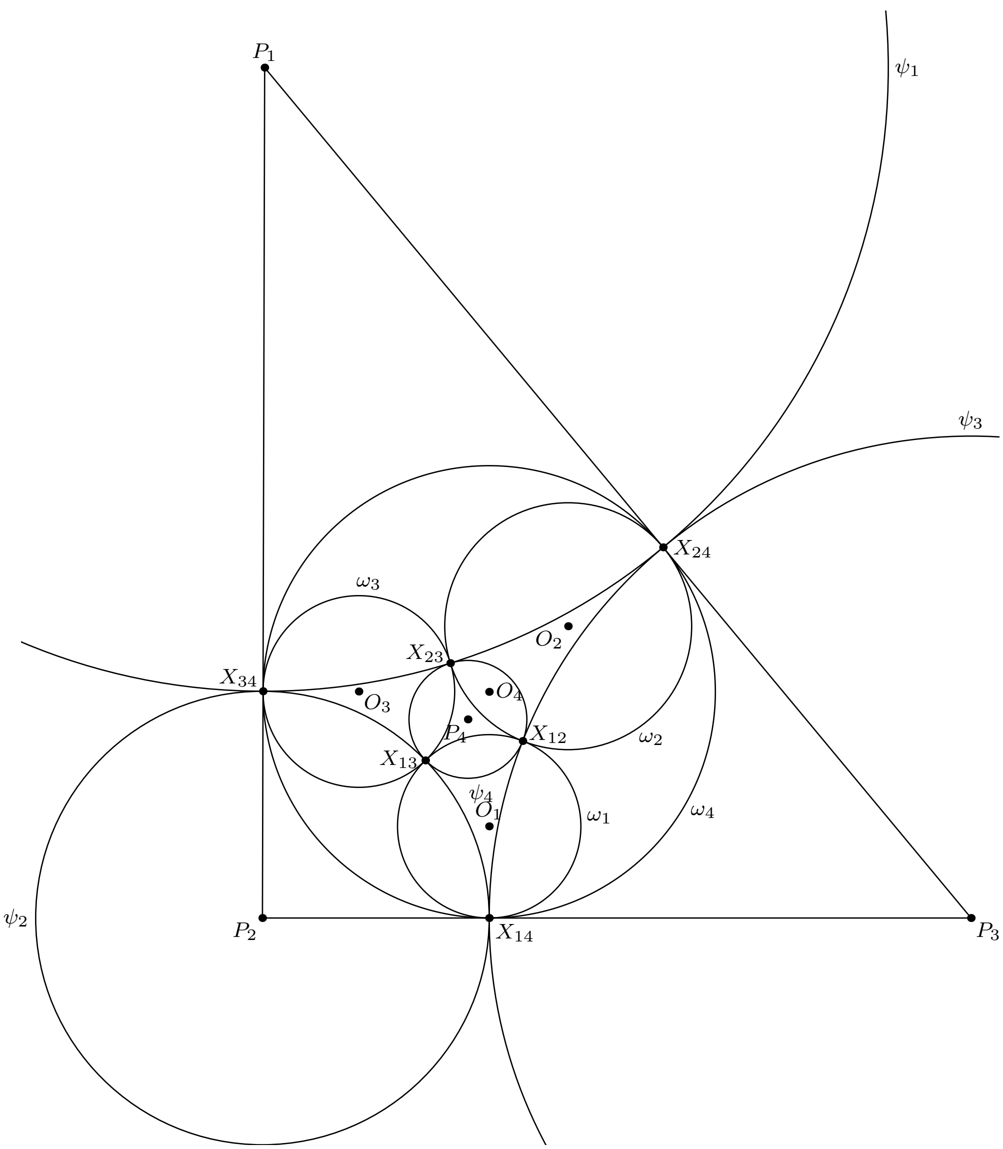
The conditions of the problem imply that P_1 is the radical center of ω_2, ω_3, ω_4, and similarly for the other P_i. If $i \neq j$, let X_{ij} be the point where ω_i intersects and is tangent to ω_j, and write t_{ij} for their common tangent line through X_{ij}. The line t_{ij} is the radical axis of ω_i, ω_j. It follows, for example, that t_{23}, t_{24}, t_{34} all pass through P_1; in general, t_{ij} passes through P_k if i, j, k are distinct.

Let us rule out several "degenerate possibilities." The O_i must be distinct since tangent circles with the same center coincide. If $i \neq j$, then O_i is inside ω_i while P_j is outside, so $O_i \neq P_j$; the fact that $O_i \neq P_i$ is posited explicitly in the problem statement.

If say X_{12} was equal to X_{13}, then $\omega_1, \omega_2, \omega_3$ would all be tangent at this common point. Any point on the corresponding tangent line would have the same power with respect to all three circles, and so the radical center P_4 would not be well-defined. If X_{12} was equal to X_{34}, then all four circles would be tangent at this common point, which would again lead to problems. Hence all of the X_{ij} are distinct.

If say P_1 were equal to P_2, then this common point would lie on the tangents to ω_3 at the three distinct points X_{13}, X_{23}, X_{34}, which is impossible. The P_i are therefore also distinct.

By the two-tangent lemma, $P_1X_{34} = P_1X_{23} = P_1X_{24}$, so X_{34}, X_{23}, X_{24} lie on a circle ψ_1 centered at P_1. Define circles ψ_2, ψ_3, ψ_4 analogously, so in general ψ_i is centered at P_i and passes through the three points X_{jk} with i, j, k distinct. We obtain the diagram below:



The radius $\overline{O_3X_{34}}$ of ω_3 is perpendicular to the tangent segment $\overline{P_1X_{34}}$, which is in turn a radius of ψ_1. Hence $\overline{O_3X_{34}}$ is tangent to ψ_1 at X_{34}, so ω_3 and ψ_1 have perpendicular tangent lines at X_{34}, i.e. they are orthogonal circles. In general, ω_i is orthogonal to ψ_j whenever $i \neq j$, and they intersect at the two points X_{ik} where $k \neq i, j$.

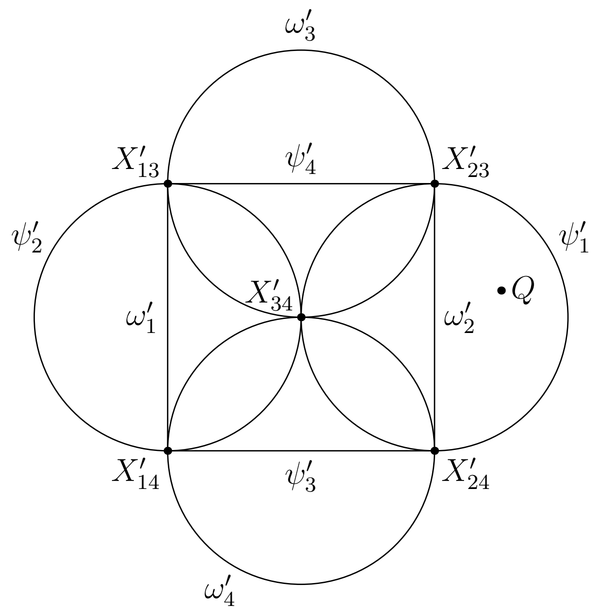
Since ψ_1 and ψ_2 both cut ω_3 (and ω_4) orthogonally at X_{34}, they are tangent at that point. In general, ψ_i and ψ_j are tangent at X_{kl}, where i, j, k, l are all distinct.

(Although we do not need this fact for the proof, it is not hard to show that the radical center of ψ_1, ψ_2, ψ_3 is O_4, and similarly for the other indices; the configuration is consequently "self-dual.")

We are now going to break some of the symmetry in this configuration by inverting with respect to a circle centered at X_{12}. The circles ω_1, ω_2, ψ_3, ψ_4 all pass through X_{12}, where ω_1, ω_2 are tangent, ψ_3 and ψ_4 are also tangent, and ω_1, ω_2 are orthogonal to ψ_3, ψ_4. These four circles therefore invert into lines, and we have $\omega_1' \parallel \omega_2' \perp \psi_3' \parallel \psi_4'$. In other words, these lines form a rectangle.

The second point (besides X_{12}) where ω_1 intersects ψ_3 is X_{14}, so X_{14}' is the vertex of the rectangle where ω_1' intersects ψ_3'. Similarly, ω_1' intersects ψ_4' at X_{13}', ω_2' intersects ψ_3' at X_{24}', and ω_2' intersects ψ_4' at X_{23}'. The rectangle is thus $X_{14}'X_{13}'X_{23}'X_{24}'$.

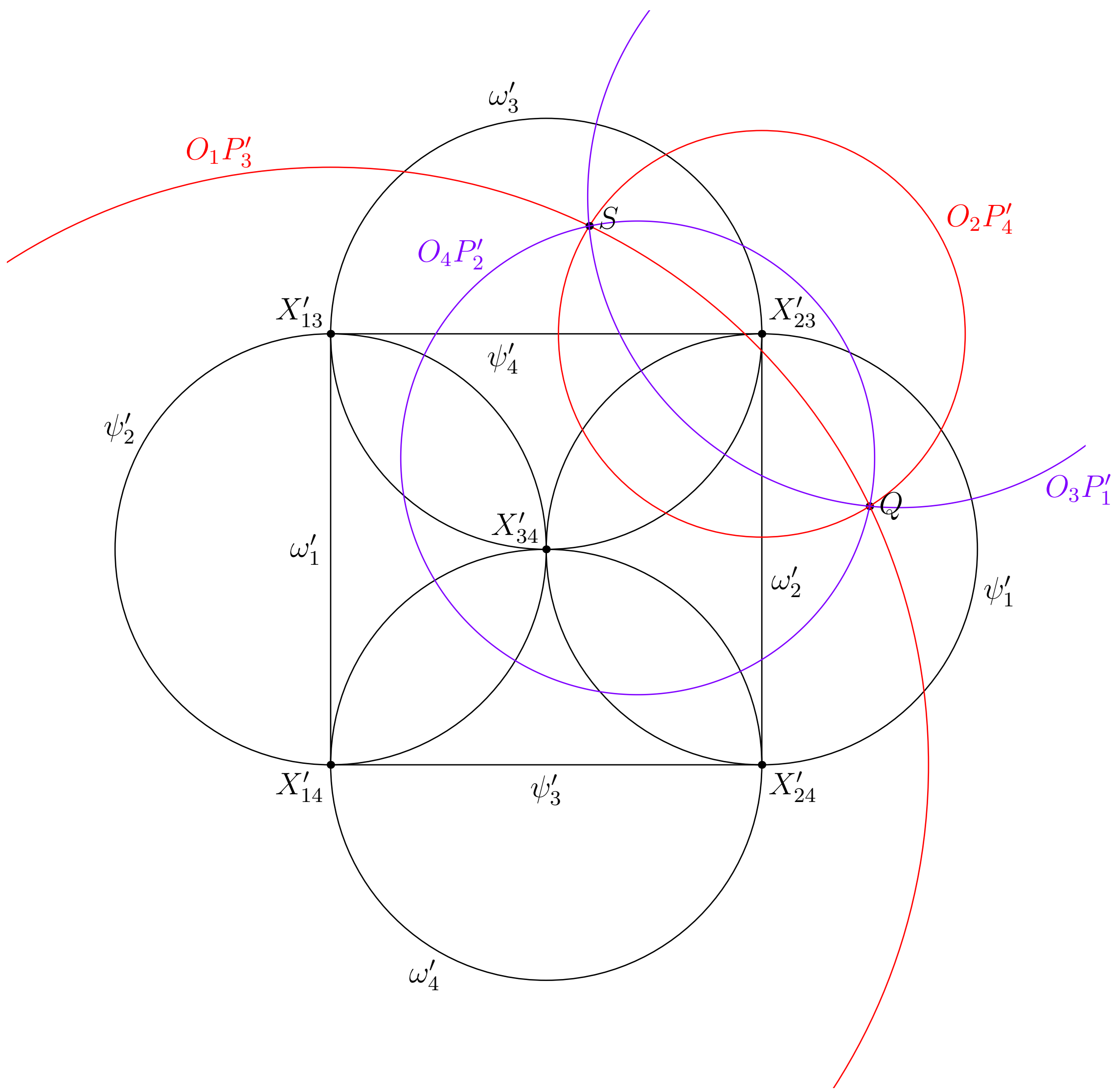
Circle ψ_2 is orthogonal to ω_1 and passes through X_{13} and X_{14}. It follows that circle ψ_2' is orthogonal to ω_1' and passes through X_{13}', X_{14}', i.e. it is the circle with diameter $\overline{X_{13}'X_{14}'}$. Similarly, ψ_1' has diameter $\overline{X_{23}'X_{24}'}$, while ω_3' has diameter $\overline{X_{14}'X_{24}'}$ and ω_4' has diameter $\overline{X_{13}'X_{23}'}$. All four of the circles ψ_1, ψ_2, ω_3, and ω_4 pass through X_{34}, where ψ_1, ψ_2 are tangent, ω_3, ω_4 are tangent, and ψ_1, ψ_2 are orthogonal to ω_3, ω_4. Thus the circles ψ_1', ψ_2', ω_3', and ω_4' pass through X_{34}', where ψ_1', ψ_2' are tangent, ω_3', ω_4' are tangent, and ψ_1', ψ_2' are orthogonal to ω_3', ω_4'. The distance between lines ω_1' and ω_2' is therefore twice the radius of circle ω_1', which is also the distance between the lines ψ_3' and ψ_4'. The rectangle in question is thus a square, and we obtain the figure below:



Of the original six points X_{ij} where the ω_i, ω_j intersected (also where the ψ_k, ψ_l intersected), four are the vertices X_{14}', X_{13}', X_{24}', X_{23}' of the square. The point X_{12}' is the inverse of the center of inversion X_{12}, and so it is at infinity. The point X_{34}' has already been identified, and it is the center of the square. The only part of the diagram that is **not** symmetrical is the center of inversion X_{12} itself, which is almost unconstrained (it is the inverse of the point at infinity, so it does not lie on any of the ω_i' or ψ_i', given that these are all inverses of circles; in particular, it is not equal to any of the X_{ij}'). In order to emphasize the special role of this point, we will call it Q (and it is labeled as such in the diagram).

A line $\overleftrightarrow{O_iP_j}$ passes through the centers of ω_i and ψ_j, and so it is orthogonal to both circles. The inverse $\overleftrightarrow{O_iP_j}'$ is then the unique circle (or line) through Q that is orthogonal to the circles/lines ω_i' and ψ_j'.

When ω_i' and ψ_j' are both lines, intersecting at one of the vertices of the square, $\overleftrightarrow{O_iP_j}'$ must be a circle centered at that vertex and passing through Q. For example $\overleftrightarrow{O_1P_3}'$ is orthogonal to ω_1' and ψ_3', and so it is the circle centered at X_{14}' passing through Q. Similarly, $\overleftrightarrow{O_2P_4}'$ is the circle centered at X_{23}' passing through Q. In the figure below, these two additional circles are illustrated in red:



Any circle is symmetric relative to a line through its center, so $\overleftrightarrow{O_2P_4}'$ and $\overleftrightarrow{O_1P_3}'$ are both symmetrical in the diagonal line $\overleftrightarrow{X_{14}'X_{23}'}$. Let S be the reflection of Q in $\overleftrightarrow{X_{14}'X_{23}'}$. Any circle that is symmetrical in $\overleftrightarrow{X'_{14}X'_{23}}$ that passes through Q also passes through S; if $S = Q$ (which happens when Q lies on $\overleftrightarrow{X'_{14}X'_{23}}$), all circles that pass through Q and are symmetrical in $\overleftrightarrow{X'_{14}X'_{23}}$ are tangent at Q. Hence all lines $\overleftrightarrow{O_iP'_j}$ such that $\overleftrightarrow{O_iP'_j}$ is symmetric across $\overleftrightarrow{X'_{14}X'_{23}}$ are concurrent at S', or, if $S = Q$, all such lines are parallel. In particular, $\overleftrightarrow{O_2P'_4}$ and $\overleftrightarrow{O_1P'_3}$ either intersect at S' or are parallel.

Consider now the circle $\overleftrightarrow{O_3P_1}'$. It is orthogonal to the two circles ω'_3 and ψ'_1, which are interchanged by reflection in $\overleftrightarrow{X'_{14}X'_{23}}$. It seems visually plausible that any circle ϕ orthogonal to these two circles is symmetric across this line. If so, then $\overleftrightarrow{O_4P_2}'$, a circle (drawn in purple) passing through Q that is orthogonal to ω'_4 and ψ'_2, must also be symmetric across $\overleftrightarrow{X'_{14}X'_{23}}$. The same will hold for the purple circle $\overleftrightarrow{O_3P_1}'$.

For this to work, we need:

Lemma 1. *Let ω be a circle and let ξ be a reflection. Suppose $\xi(\omega) \neq \omega$. Then any circle ϕ orthogonal to ω and $\xi(\omega)$ satisfies $\xi(\phi) = \phi$.*

This can be proven in a number of ways; we give a proof that involves ideas that will be useful later. Recall that any two distinct generalized circles (i.e. circles or lines) ω_1, ω_2 are part of a unique **pencil of coaxal (generalized) circles**. The theory below holds with certain modifications in the "aberrant" cases where ω_1, ω_2 are concentric or where both are lines; for the sake of simplicity, we exclude these cases (which are not relevant for the proof here). The pencil then has the following properties (cf. *Geometry Revisited*, sections 2.3 (p. 55), 5.7 (p. 120)):

(1) All circles in the pencil have centers along a fixed line ("axis," whence the name "coaxal"). There is a unique line in the pencil, and it is perpendicular to the axis.

(2) If ω_1, ω_2 are tangent, say at P, with ℓ the common tangent at P, then the pencil consists of all circles tangent to ℓ at P, together with ℓ itself.

(3) If ω_1, ω_2 intersect in two points, say P and Q, then the pencil consists of all generalized circles through P and Q (including the line $\overleftrightarrow{PQ}$).

(4) If ω_1, ω_2 do not intersect, then all circles in the pencil are disjoint, and the line in the pencil is the radical axis of any two of these circles.

(5) The set of all generalized circles orthogonal to ω_1, ω_2 forms another pencil of coaxal circles. Each ϕ in this pencil is orthogonal not only to ω_1 and ω_2, but to all circles in the original pencil. The axis of the new pencil is perpendicular to the axis of the old pencil. Moreover, if the original pencil is of type (2), the orthogonal pencil is also of type (2), while if it is of type (3), the orthogonal pencil is of type (4) and vice versa.

The lemma can now be proven as follows:

Proof. Consider the pencil of coaxal circles determined by ω and $\xi(\omega)$. We claim that the line of reflection ℓ is in this pencil. In fact, this is clear in every case: if ω and $\xi(\omega)$ intersect in two points, then ℓ passes through these two points; if they are tangent, then ℓ is the relevant common tangent; if they do not intersect, then ℓ is their radical axis.

Since any circle ϕ orthogonal to ω and $\xi(\omega)$ is orthogonal to every circle in their pencil, it follows that ϕ is also orthogonal to ℓ. Hence ϕ is symmetric about ℓ, i.e. $\xi(\phi) = \phi$. □

We have therefore proven that the lines

$$\overleftrightarrow{O_1P_3}, \overleftrightarrow{O_2P_4}, \overleftrightarrow{O_3P_1}, \overleftrightarrow{O_4P_2}$$

are concurrent (or possibly parallel). By symmetry in the subscripts, the same holds for the lines

$$\overleftrightarrow{O_1P_4}, \overleftrightarrow{O_2P_3}, \overleftrightarrow{O_3P_2}, \overleftrightarrow{O_4P_1}$$

and

$$\overleftrightarrow{O_1P_2}, \overleftrightarrow{O_2P_1}, \overleftrightarrow{O_3P_4}, \overleftrightarrow{O_4P_3}.$$

The remaining set of concurrent/parallel lines must therefore be

$$\overleftrightarrow{O_1P_1}, \overleftrightarrow{O_2P_2}, \overleftrightarrow{O_3P_3}, \overleftrightarrow{O_4P_4}.$$

To figure out what is going on in this last case, let us try to figure out where the second and third sets intersect. The case of the circles

$$\overleftrightarrow{O_1P_4}', \overleftrightarrow{O_2P_3}', \overleftrightarrow{O_3P_2}', \overleftrightarrow{O_4P_1}'$$

is analogous to the first set we considered. The circles $\overleftrightarrow{O_1P_4}'$ and $\overleftrightarrow{O_2P_3}'$ intersect at the point T that is the reflection of Q across the other diagonal $X'_{13}X'_{24}$ of the square. Lemma 1 above shows that the circles $\overleftrightarrow{O_3P_2}'$, $\overleftrightarrow{O_4P_1}$ also pass through T. If $T = Q$, then all of the circles are tangent at Q.

The circles

$$\overleftrightarrow{O_1P_2}', \overleftrightarrow{O_2P_1}', \overleftrightarrow{O_3P_4}', \overleftrightarrow{O_4P_3}'$$

are more interesting. $\overleftrightarrow{O_1P_2}'$ is a circle passing through Q that is orthogonal to the line ω'_1 and the circle ψ'_2. Let χ be the circle (not pictured) centered at X'_{34} and passing through the vertices X'_{13}, X'_{23}, X'_{14}, X'_{24} of the square. Inversion in χ takes ω'_1 to ψ'_2. Similarly, $\overleftrightarrow{O_2P_1}'$ is orthogonal to the line ω'_2 and the circle ψ'_1, and inversion in χ takes ω'_2 to ψ'_1. Analogous statements hold for $\overleftrightarrow{O_3P_4}'$ and $\overleftrightarrow{O_4P_3}'$. Considering inversion as being "like" reflection, we hypothesize that all four circles pass through the point U that is the inverse of Q with respect to χ (and if $U = Q$, which occurs if Q lies on χ, we hypothesize that all four circles are tangent at Q). If the four circles are orthogonal to χ, then all of this will hold, so we claim:

Lemma 2. *Let ω be a circle and let ξ be inversion with respect to some (nonconcentric) circle. Suppose $\xi(\omega) \neq \omega$. Then any circle ϕ orthogonal to ω and $\xi(\omega)$ satisfies $\xi(\phi) = \phi$.*

This result holds and the proof is not difficult, but since we have already proven that the lines in question are concurrent/parallel, we leave the proof to the reader.

Turning to the (anomalous) fourth set of lines, the corresponding circles are all analogous to $\overleftrightarrow{O_1P_1}'$, which is orthogonal to the line ω'_1 and to the circle ψ'_1. The inverse of ω'_1 is ψ'_2, so ψ'_1 is the result of inversion in χ **followed by a half-turn** about the center of inversion X'_{34}. In fact, for all i, the generalized circles ω'_i and ψ'_i are interchanged by this combination of inversion and half-turn. Following the pattern seen in the other cases, we hypothesize that $\overleftrightarrow{O_1P_1}'$ typically passes through the point V obtained from Q by inverting to U and then performing a half-turn about X'_{34}. No point in the inversive plane is mapped to itself by an inversion followed by a half-turn in the center of inversion, so the case $V = Q$ does not arise. We therefore replace "typically" by "always," hypothesizing that the lines

$$\overleftrightarrow{O_1P_1}, \overleftrightarrow{O_2P_2}, \overleftrightarrow{O_3P_3}, \overleftrightarrow{O_4P_4}$$

are unconditionally concurrent at V' (they are never parallel).

Here the lemma we wish to prove is:

Lemma 3. *Let ω be a circle and let ξ be inversion with respect to some (nonconcentric) circle, followed by a half-turn about the center of the circle of inversion. Suppose $\xi(\omega) \neq \omega$. Then any circle ϕ orthogonal to ω and $\xi(\omega)$ satisfies $\xi(\phi) = \phi$.*

This lemma as well can be proven in a variety of ways. Our argument involves an auxiliary lemma that might be helpful in other contexts:

Lemma 4. *Let ξ be either inversion or inversion followed by a half-turn about the center of inversion. Suppose P is a point such that $\xi(P) \neq P$. Then any circle α passing through P and $\xi(P)$ satisfies $\xi(\alpha) = \alpha$.*

Proof. Let the circle of inversion have center O and radius r. If $P = O$, then α is a line through the origin and the result is obvious, so suppose $P \neq O$. The line ℓ through O passing through P also passes through $\xi(P)$. By the definition of inversion, $OP \cdot O\xi(P)$ is either r^2 (if ξ is inversion) or $-r^2$ (if ξ is inversion followed by a half-turn). Take any other line through O, intersecting α at two points, say M and N. By power of a point, $OM \cdot ON = OP \cdot O\xi(P) = \pm r^2$, so $N = \xi(M)$. Hence $\xi(\alpha)$ has infinitely many points in common with α. Three points suffice to determine a circle, so $\xi(\alpha) = \alpha$. □

Here is our proof of Lemma 3:

Proof. Let the circle of inversion have center O. If ω and $\xi(\omega)$ intersected in a point P, then $\xi(P)$ would be a point of $\xi(\xi(\omega)) = \omega$. Inversion followed by a half-turn about the center of inversion has no fixed points, so P and $\xi(P)$ would be distinct points of ω. Lemma 4 would then imply $\xi(\omega) = \omega$, contrary to hypothesis. Hence ω and $\xi(\omega)$ are disjoint.

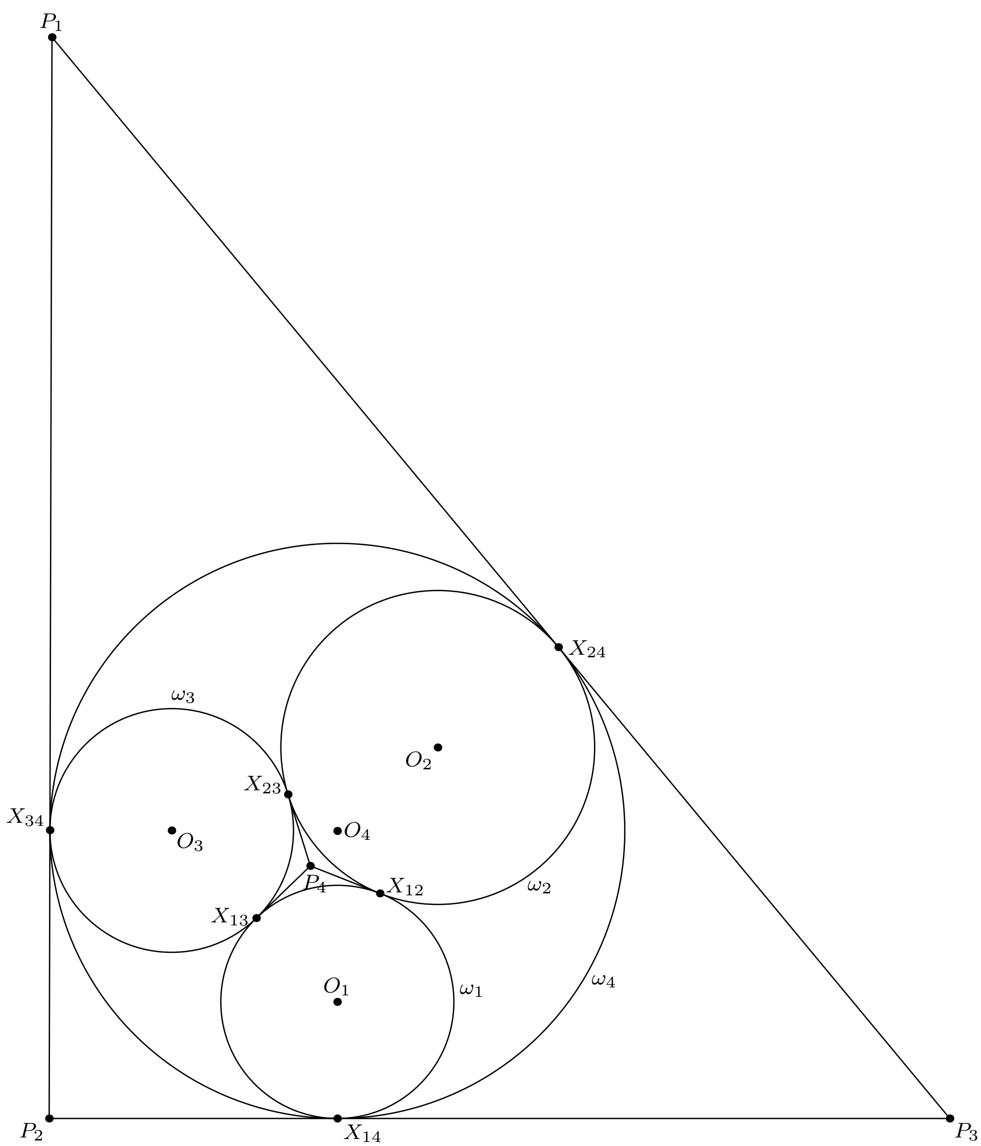
Any circles related by inversion in O or a half-turn in O are related by a homothety from O. Since $\xi(\omega) \neq \omega$, this homothety is not the identity. Hence $\xi(\omega)$ only has the same center as ω if ω has center O. The hypothesis excludes this possibility, so ω and $\xi(\omega)$ are non-concentric.

It follows that ω and $\xi(\omega)$ determine a pencil of disjoint coaxal circles. Any circle ϕ orthogonal to ω and $\xi(\omega)$ lies in the corresponding orthogonal pencil, which consists of circles passing through distinct fixed points K and L. If ϕ is orthogonal to ω and $\xi(\omega)$, then $\xi(\phi)$ is orthogonal to $\xi(\omega)$

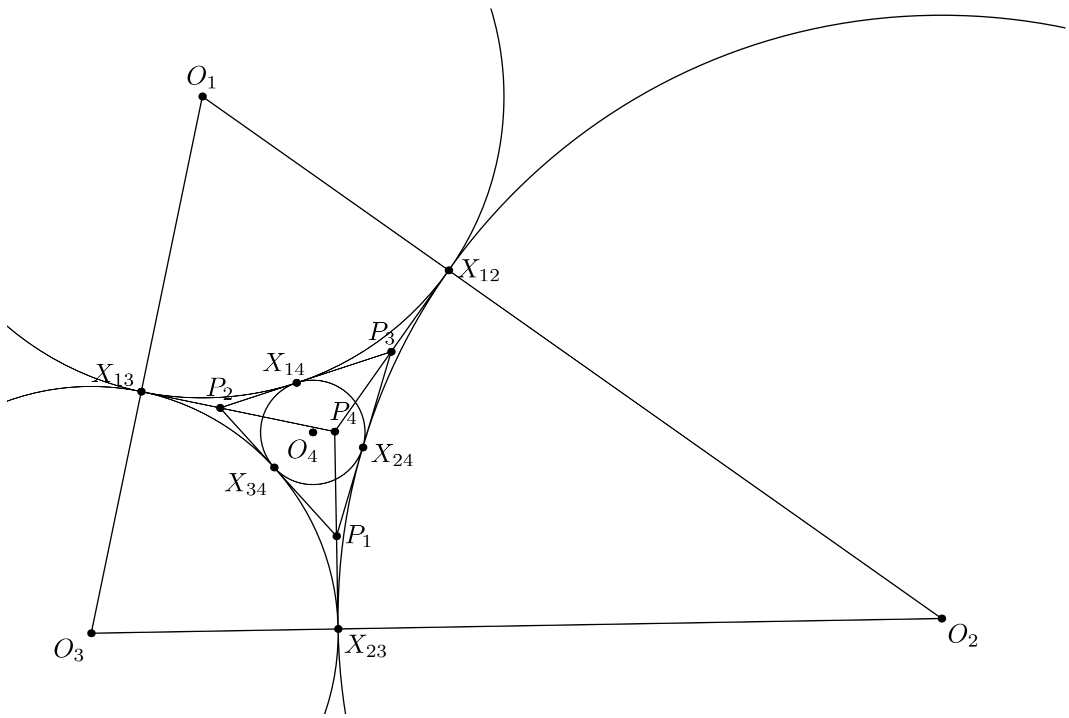
and $\xi(\xi(\omega)) = \omega$, so $\xi(\phi)$ is also in the orthogonal pencil. It follows that $\xi(K)$ is common to every circle in the orthogonal pencil, so it is either K or L. Using again the fact that ξ has no fixed points, $\xi(K) = L$. Therefore ϕ passes through distinct points K and $\xi(K)$, so by Lemma 4, $\xi(\phi) = \phi$. □

The proof is now complete.

Solution 2: For reference, we include the original configuration again below, and we similarly exclude degenerate possibilities.



In the diagram above, ω_4 contains the other three circles. In general, if i, j, k, l are distinct and ω_i contains ω_j, then since ω_k and ω_l are tangent to ω_j, they must also lie inside ω_i. The diagram above is therefore essentially the only possible configuration in which one circle contains another in its interior. Otherwise all four circles are externally tangent to each other, and we get a diagram like the one below, in which three circles are externally tangent and a fourth circle is externally tangent to all three of them and lies within the bounded region determined by the first three circles.

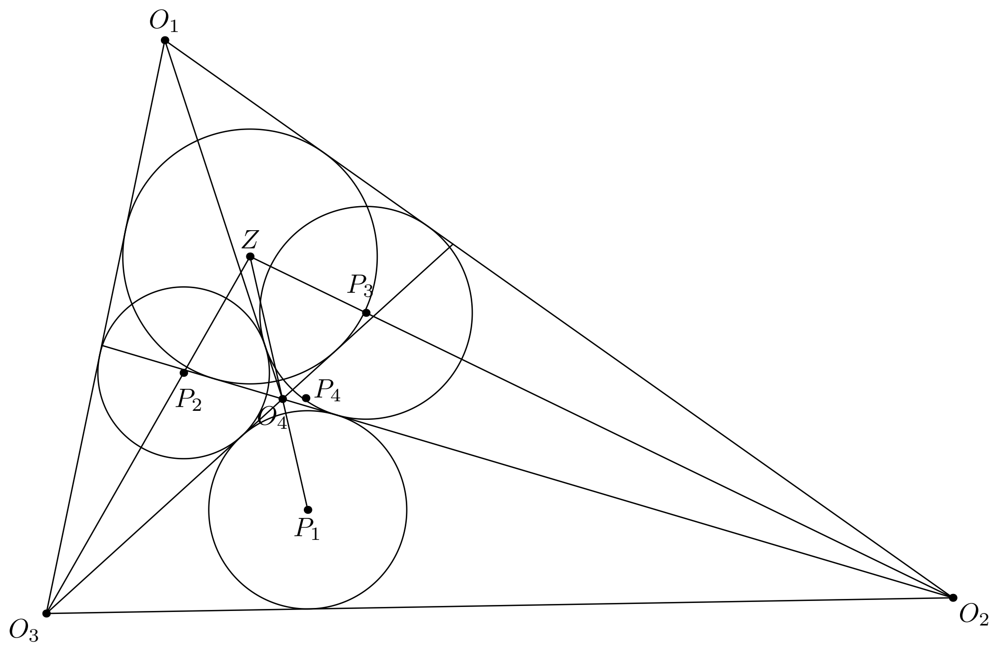


Consider first this new diagram. Since $\overline{P_1X_{34}}$ is tangent to ω_3 and ω_4, it is perpendicular to the line $\overleftrightarrow{O_3O_4}$ joining their centers; clearly X_{34} is between O_3 and O_4 along this line. Similarly, $\overline{P_1X_{24}}$ is perpendicular to $\overleftrightarrow{O_2O_4}$ and X_{24} lies between O_2 and O_4. Similarly again, $\overline{P_1X_{23}}$ is perpendicular to $\overleftrightarrow{O_2O_3}$ and X_{23} lies between O_2 and O_3. It follows that P_1 is equidistant from the lines $\overleftrightarrow{O_3O_4}$, $\overleftrightarrow{O_2O_4}$, $\overleftrightarrow{O_2O_3}$, and lies within the triangle they form. It is therefore the incenter of $\triangle O_2O_3O_4$.

In the same way P_2 is the incenter of $\triangle O_1O_3O_4$ and P_3 is the incenter of $\triangle O_1O_2O_4$. The point P_4 is superficially a bit different, but again it is equidistant from the lines $\overleftrightarrow{O_1O_2}$, $\overleftrightarrow{O_1O_3}$, $\overleftrightarrow{O_2O_3}$ and lies within $\triangle O_1O_2O_3$, so it is the incenter of $\triangle O_1O_2O_3$.

Let us now look at the original diagram. Here the same logic shows that P_4 is the incenter of $\triangle O_1O_2O_3$. The case of P_1 is however a bit different. It is again equidistant from $\overleftrightarrow{O_3O_4}$, $\overleftrightarrow{O_2O_4}$, $\overleftrightarrow{O_2O_3}$, but here the feet of the perpendiculars to the sides of $\triangle O_2O_3O_4$ are situated as follows: X_{23} lies between O_3 and O_2, but O_3 is between X_{34} and O_4, while O_2 is between O_4 and X_{24}. Consequently here P_1 is not the incenter of $\triangle O_2O_3O_4$, but one of the excenters, and specifically it is the O_4-excenter. In the same way, P_2 is the O_4-excenter of $\triangle O_1O_3O_4$ and P_3 is the O_4-excenter of $\triangle O_1O_2O_4$.

Write now r_i for the radius of circle ω_i. Suppose no circle contains another. Consider the concave quadrilateral $O_1O_2O_4O_3$. Observe that $O_1O_3+O_2O_4 = (r_1+r_3)+(r_2+r_4)$, while $O_1O_2+O_3O_4 = (r_1+r_2)+(r_3+r_4)$. The opposite sides of $O_1O_2O_4O_3$ therefore have the same sum. This looks like Pitot's theorem, and in fact Pitot's theorem applies to concave quadrilaterals as well. It follows that there exists a circle γ inside $O_1O_2O_4O_3$ that is tangent to $\overline{O_1O_2}$ and $\overline{O_1O_3}$, and also tangent to the extensions of $\overline{O_2O_4}$ and $\overline{O_3O_4}$. Write Z for the center of γ. This new circle is shown in the diagram below.



P_4 is the incenter of $\triangle O_1O_2O_3$, while Z is the center of γ, which is inside and tangent to $\angle O_2O_1O_3$. Hence Z and P_4 both lie on the angle bisector of $\angle O_2O_1O_3$, so O_1, P_4, Z are collinear. Point P_1 is the incenter of $\triangle O_2O_3O_4$, while Z is the center of a circle that is tangent to the extensions of $\overline{O_2O_4}$ and $\overline{O_3O_4}$ past O_4. It follows that P_1 and Z both lie on the (internal) angle bisector of $\angle O_2O_4O_3$ with P_1 lying on the ray inside the angle and Z lying on the opposite ray.

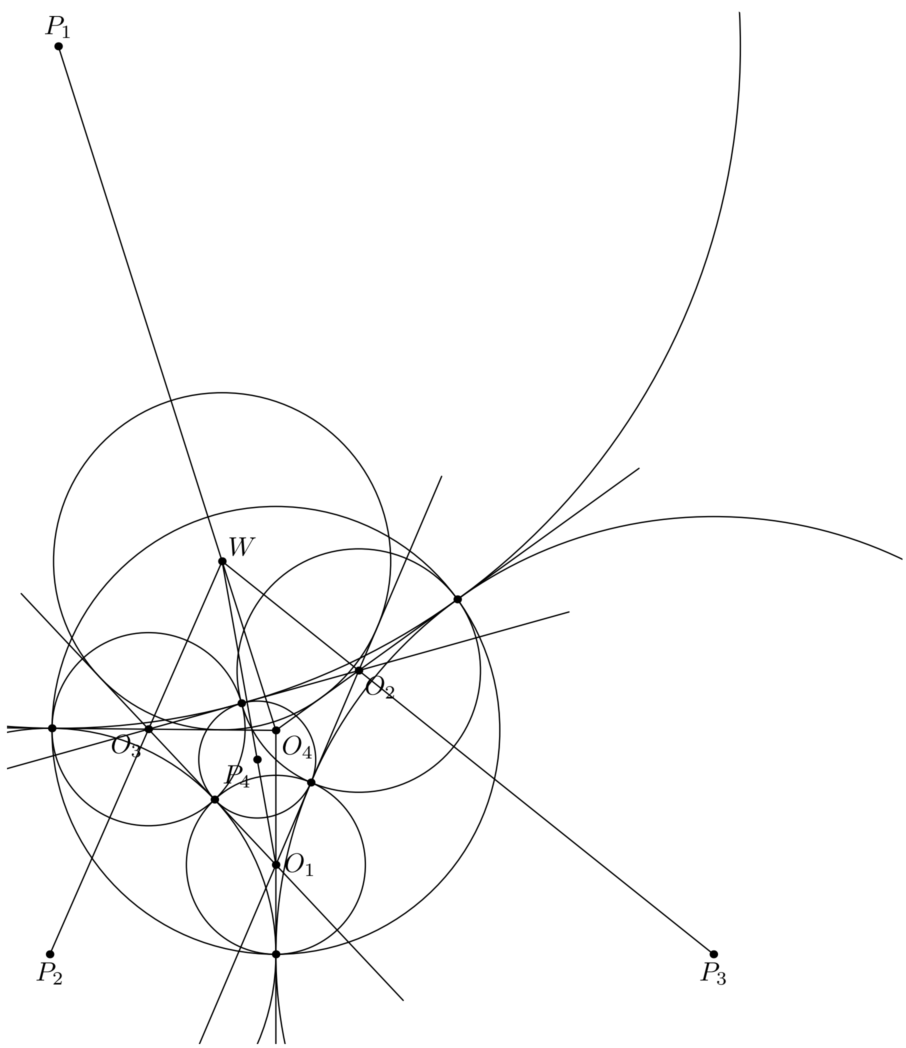
P_3 is the incenter of $\triangle O_1O_2O_4$. Here γ is tangent to $\overline{O_2O_1}$ and to the extension of $\overline{O_2O_4}$ past O_4. Hence P_3 and Z both lie on the angle bisector of $\angle O_1O_2O_4$. The case of P_2 is completely analogous—the same reasoning shows that P_2 and Z both lie on the angle bisector of $O_1O_3O_4$.

Putting all of these facts together, we see that Z lies on the four lines

$$\overleftrightarrow{O_1P_4}, \overleftrightarrow{O_2P_3}, \overleftrightarrow{O_3P_2}, \overleftrightarrow{O_4P_1},$$

i.e. these lines are concurrent.

Let us try to adapt this argument to the other configuration. Here we have $O_1O_2 = r_1 + r_2$, $O_1O_3 = r_1 + r_3$, $O_2O_3 = r_2 + r_3$, but $O_1O_4 = r_4 - r_1$, $O_2O_4 = r_4 - r_2$, and $O_3O_4 = r_4 - r_3$. Here opposite sides are in general not equal, but we note that $O_4O_2 + O_2O_1 = (r_4 - r_2) + (r_2 + r_1)$, while $O_4O_3 + O_3O_1 = (r_4 - r_3) + (r_3 + r_1)$. Hence $O_4O_2 + O_2O_1 = O_4O_3 + O_3O_1$. There is a version of Pitot's theorem that applies in this case as well—it shows that there exists a circle δ tangent to the extended four sides of quadrilateral $O_1O_2O_4O_3$, but now the circle is *outside* the quadrilateral. The circle lies either within $\angle O_2O_4O_3$ or the angle vertical to it; it lies either within $\angle O_3O_1O_2$ or the angle vertical to it (in the diagram below, it lies in $\angle O_3O_1O_2$, and within the outside portion of reflex angle $O_2O_4O_3$. It instead lies in the angle exterior to $\angle O_1O_2O_4$ and the angle exterior to $\angle O_4O_3O_1$.



We can now reason similarly to before. Denote the center of δ as W. Then P_4 is the incenter of $\triangle O_1O_2O_3$ and so lies on the internal angle bisector of $\angle O_3O_1O_2$, while W lies on the internal angle bisector of $\angle O_3O_1O_2$ or possibly on the opposite ray. Hence O_1, P_4, W are collinear. Similarly, P_1 is the O_4-excenter of $\triangle O_2O_3O_4$, and so it lies on the internal angle bisector of $\angle O_2O_4O_3$, while W lies on the internal angle bisector of $\angle O_2O_4O_3$ or on its extension; thus O_4, P_1, W are collinear. On the other hand, P_2 is the O_4-excenter of $\triangle O_1O_3O_4$, so it lies on one of the external angle bisectors of $\angle O_4O_3O_1$, and W also lies on one of the external angle bisectors of this angle. Hence O_3, P_2, W are collinear. Completely analogous logic shows that O_2, P_3, W are collinear.

It follows that

$$\overleftrightarrow{O_1P_4}, \overleftrightarrow{O_2P_3}, \overleftrightarrow{O_3P_2}, \overleftrightarrow{O_4P_1}$$

are concurrent in both configurations.

By symmetry in the indices, the set of lines

$$\overleftrightarrow{O_1P_3}, \overleftrightarrow{O_2P_4}, \overleftrightarrow{O_3P_1}, \overleftrightarrow{O_4P_2}$$

are concurrent, as are

$$\overleftrightarrow{O_1P_2}, \overleftrightarrow{O_2P_1}, \overleftrightarrow{O_3P_4}, \overleftrightarrow{O_4P_3}.$$

The wording of the problem now implies that the four lines

$$\overleftrightarrow{O_1P_1}, \overleftrightarrow{O_2P_2}, \overleftrightarrow{O_3P_3}, \overleftrightarrow{O_4P_4}$$

are the last concurrent or parallel set.

In this approach, it is not obvious how to prove that this last set of lines are concurrent directly, but we will show that their concurrence follows (under appropriate stipulations) from the concurrence of the other three sets. Observe that if we can do this, it will also be possible to finish the inversion-based solution in a different way, since there as well the concurrence of the first three sets is proven and then the concurrence of the fourth set requires more complicated reasoning.

The statement we are trying to prove involves only points and lines and the ways that they intersect. It is therefore a statement in projective geometry *par excellence*, and we will establish it as such:

Lemma 1. *Let $O_1, O_2, O_3, O_4, P_1, P_2, P_3, P_4$ be eight distinct points. Suppose that the sets*

$$\overleftrightarrow{O_1P_3}, \overleftrightarrow{O_2P_4}, \overleftrightarrow{O_3P_1}, \overleftrightarrow{O_4P_2}$$

and

$$\overleftrightarrow{O_1P_4}, \overleftrightarrow{O_2P_3}, \overleftrightarrow{O_3P_2}, \overleftrightarrow{O_4P_1}$$

and

$$\overleftrightarrow{O_1P_2}, \overleftrightarrow{O_2P_1}, \overleftrightarrow{O_3P_4}, \overleftrightarrow{O_4P_3}.$$

are each concurrent (or possibly parallel). Suppose also that if i, j, k are distinct, then O_i, O_j, P_k are not collinear, and P_i, P_j, O_k are also not collinear. Then the lines

$$\overleftrightarrow{O_1P_1}, \overleftrightarrow{O_2P_2}, \overleftrightarrow{O_3P_3}, \overleftrightarrow{O_4P_4}.$$

are concurrent (or possibly parallel).

Proof. Work within the projective plane, so "parallel" is a special case of "concurrent." Let the points at which the first three sets concur be X, Y, Z respectively. Consider $\triangle O_1P_3O_2$ and $\triangle P_2O_4P_1$. By assumption, these are nondegenerate triangles. The lines $\overleftrightarrow{O_1P_2}, \overleftrightarrow{P_3O_4}, \overleftrightarrow{O_2P_1}$ intersect at Z, so the triangles are in perspective centrally. By Desargues' theorem, they are in perspective axially, so the three points $X = \overleftrightarrow{O_1P_3} \cap \overleftrightarrow{P_2O_4}, Y = \overleftrightarrow{P_3O_2} \cap \overleftrightarrow{O_4P_1}$ and $T := \overleftrightarrow{O_1O_2} \cap \overleftrightarrow{P_2P_1}$ are collinear. We can equally well say $X = \overleftrightarrow{O_1P_3} \cap \overleftrightarrow{P_1O_3}, Y = \overleftrightarrow{P_3O_2} \cap \overleftrightarrow{O_3P_2}$, and $T := \overleftrightarrow{O_1O_2} \cap \overleftrightarrow{P_1P_2}$ are collinear, i.e. $\triangle O_1P_3O_2$ and $\triangle P_1O_3P_2$ are in perspective axially. By Desargues' theorem in the opposite direction, they are in perspective centrally, so $\overleftrightarrow{O_1P_1}, \overleftrightarrow{O_3P_3}, \overleftrightarrow{O_2, P_2}$ are concurrent in a point W. By symmetry in the indices (or making an analogous argument with respect to $\triangle O_1P_4O_2$ and $\triangle P_1O_4P_2$), $\overleftrightarrow{O_4P_4}$ also passes through W. □

It is actually possible to prove the result with weaker hypotheses: it is enough to assume that the O_i, O_j, P_k are not collinear, and it is also enough to assume that the P_i, P_j, O_k are not collinear. However, if one assumes neither collinearity hypothesis, the conclusion may fail. The proof that weaker hypotheses suffice is a bit technical, and in practice it is easier to use the stronger hypotheses, given that in the case at hand they are easily shown to hold:

Lemma 2. *The noncollinearity hypotheses hold in the problem above.*

Proof. Line $\overleftrightarrow{P_1P_2}$ is externally tangent to ω_3, so it cannot pass through O_3, the center of ω_3. Similarly O_i, P_j, P_k are noncollinear for all distinct i, j, k.

Suppose e.g. that P_3 is collinear with O_1, O_2. Since P_3 is on the common tangent to ω_1 and ω_2 through X_{12}, we would have to have $P_3 = X_{12}$. But X_{12} has power zero with respect to ω_1 and ω_2, so if it is the radical center of ω_2, ω_3, ω_4, then ω_4 also passes through X_{12}. As explained in the main solution, this is impossible—these three circles cannot be tangent at the same point because then P_3 would not be unique. □