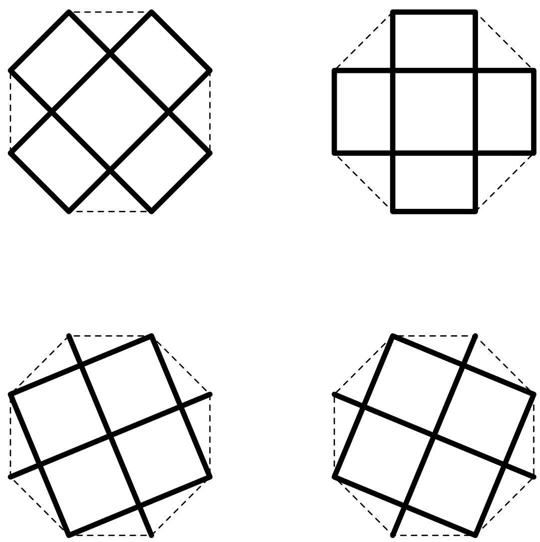
1. There are 12 squares.

   Since an octagon has 8 vertices, there are $\binom{8}{2} = 28$ edges and interior diagonals. Define two lines (edges or diagonals) to be *compatible* if they are either parallel or perpendicular. That is, they are the same up to translations and 90-degree rotations.

   Notice that any square that is made in the octagon is made of four compatible lines. That is, each side of the square is parallel or perpendicular to each other side of the square. Therefore, to find all squares, we can look for equivalence classes of compatible lines (all the lines in a set equivalent to each other) and we only need to consider squares within each equivalence class.

   There are four such equivalence classes, and we can draw them as follows:



   The top two equivalence classes have 1 square each, and the bottom two equivalence classes have 5 squares each. Thus the answer is
   $$1 + 1 + 5 + 5 = 12.$$

2. Using polynomial division, we can write $p(x) = (x^2 + x)A(x) + bx + c$ where $A(x)$ is a polynomial with real coefficients and b and c are constants.

   Then we find
   $$\left\lfloor \frac{p(x)}{x} \right\rfloor = (x+1)A(x) + b \quad \text{and} \quad \left\lfloor \frac{p(x)}{x+1} \right\rfloor = xA(x) + b,$$
   so adding these, $(2x+1)A(x) + 2b = x^2$.

   Plugging in $x = -\frac{1}{2}$, it follows that $2b = \frac{1}{4}$, so $b = \frac{1}{8}$. Therefore, $(2x+1)A(x) + \frac{1}{4} = x^2$, or $2(x + \frac{1}{2})A(x) = (x + \frac{1}{2})(x - \frac{1}{2})$. Hence, $A(x) = \frac{1}{2}x - \frac{1}{4}$. Any value of c works.

   Plugging these back in for $p(x)$ and expanding, the answer we get is
   $$p(x) = \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2 - \tfrac{1}{8}x + c \quad \text{for any } c \in \mathbb{R}.$$

3. We claim that Paris has the winning strategy as follows: he can win by picking apples on each turn and stealing apples only if it leads to an immediate win.

   Define
$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
   to be the n^{th} triangular number, and note that $T_{49} = 1225$, the required number of apples to win. Since he picks one more apple each time he picks, and after n picks (if no apples are stolen) he ends up with T_n apples, Menelaus cannot win until at least the 49^{th} turn.

   We split the game into three possible cases.

   - First, suppose that Menelaus does not spoil apples in the first 35 turns. On Paris's 35th turn, he can steal and win, adding Menalaus's T_{35} apples to his current T_{34} apples for a total of
$$\frac{34 \cdot 35}{2} + \frac{35 \cdot 36}{2} = 35^2 = 1225.$$
   - Second, suppose that Menelaus spoils Paris's apples once within the first 35 turns, but does not spoil again before turn 50. Then, after his 49^{th} turn, Menelaus has T_{48} apples and Paris has
$$p \geq 48 + 47 + \cdots + 35 > 49.$$
     Once again, Paris can steal immediately to obtain at least $T_{48} + 49 = T_{49}$ apples and win.
   - Finally, suppose that after turn 49, Menelaus has spoiled Paris's apples at least twice. Let Menelaus pick apples for the 48^{th} time on turn N. (If such a turn N does not exist, then Paris wins after picking at most 1225 times, so the proof is done.) Since Menelaus has spoiled twice, we have $N \geq 50$. After Menelaus picks on turn N, Paris has picked every turn and thus has at least the $N-1$ apples from turn $N-1$ unspoiled in his basket. Then Paris can steal on turn N, achieving
$$p \geq N - 1 + T_{48} \geq 49 + T_{48} = T_{49} = 1225$$
     apples in his basket and winning the game.

   Since these three cases exhaust all possibilities under Paris's given strategy, Paris has the winning strategy.

4. **Solution 1:** The answer is 3375. Let $S(N)$ denote the split-sum of N. We begin with two lemmas.

   Lemma 1: For any integer N with $2d$ or $2d-1$ digits, $S(N) \equiv N \pmod{10^d - 1}$.

   Proof. Let a make up the first d or $d-1$ digits of N and b make up the last d digits. Then
$$S(N) = a + b \equiv 10^d \cdot a + b = N \pmod{10^d - 1}.$$
   Now suppose that m is some positive integer with k digits. Then m^2 has either $2k-1$ or $2k$ digits, so $S(m^2) \equiv m^2 \pmod{10^k - 1}$. □

Lemma 2: Let m be a $k-$digit positive integer. If $m^2 \equiv m \pmod{10^k - 1}$, then $S(m^2) = m$.

Proof. By Lemma 1, $S(m^2) \equiv m^2 \pmod{10^k - 1}$.

Let $S(m^2) = a + b$ following the same notation as in the proof of Lemma 1. We have $m^2 < 10^k \cdot m$, which implies $a < m$. Since $b \leq 10^k - 1$, it follows that

$$S(m^2) = a + b < m + 10^k - 1.$$

Moreover,

$$S(m^2) > 0 \geq m - (10^k - 1).$$

Putting these together, if $S(m^2) \equiv m \pmod{10^k - 1}$ then $S(m^2)$ must equal m. □

Now, suppose that the digits of m are identical; then

$$m = \ell \cdot \frac{10^k - 1}{9}$$

for some $1 \leq \ell \leq 9$. Note that $\frac{10^k-1}{9}$ is a string of k $1'$s and so is divisible by 3 exactly when k is divisible by 3.

If $3 \nmid k$ then 9 is prime to $\frac{10^k-1}{9}$ so the Chinese Remainder Theorem means we need exactly

$$m^2 \equiv m \left(\bmod \frac{10^k - 1}{9}\right)$$
$$m^2 \equiv m \pmod 9.$$

The first condition is trivial because $\frac{10^k-1}{9} \mid m$. The second condition is satisfied when $m \equiv 0, 1 \pmod 9$. But $m \equiv k\ell \pmod 9$ so we need either $\ell = 9$ or ℓ to be the inverse of $k \pmod 9$, which is unique since $3 \nmid k$.

If $3 \mid k$ then we already know $\ell = 9$ works, so suppose $\ell < 9$. Using the Chinese Remainder Theorem, we need $m^2 \equiv m \pmod{3^n}$ where $n = v_3(10^k - 1)$. But $1 \leq v_3(m) < n$ so this is impossible, as $v_3(m) < v_3(m^2)$. Thus we only have $\ell = 9$.

In total we have the 2025 values for $\ell = 9$ $(9, 99, 999, \dots)$ and the $2025 \cdot 2/3 = 1350$ values when $3 \nmid k$ $(1, 55, 7777, 22222, \dots)$, for 3375 in total.

Solution 2: Let d be the common digit and n the number of digits of m, so that $m = d\frac{10^n-1}{9}$. Noting that $10^{2n-2} \leq m^2 < 10^{2n}$, m^2 has $2n - 1$ or $2n$ digits, so the split-sum of m^2 will have n digits in the second part. Thus, writing

$$m^2 = 10^n q + r$$

where $0 \leq r < 10^n$, we find that the condition holds iff $q + r = m$. Substituting $r = m^2 - 10^n q$, we rewrite this equation as

$$q + (m^2 - 10^n q) = m$$
$$\iff m(m - 1) = q(10^n - 1)$$

Finally, substituting $m = d\frac{10^n-1}{9}$, we get

$$d\frac{10^n-1}{9}\left(d\frac{10^n-1}{9}-1\right) = q(10^n-1)$$
$$\iff \underbrace{\frac{d}{9}\left(d\frac{10^n-1}{9}-1\right)}_{\text{E}} = q.$$

To recap, we have shown that m satisfies the conditions iff the expression E (above) is equal to the value of m^2 prior to the last n digits, i.e. $E = \left\lfloor \frac{m^2}{10^n} \right\rfloor$. In general, $y = \lfloor x \rfloor$ iff y is an integer and $x - y \in [0, 1)$. So we need to solve for when (i) E is an integer and (ii) $\frac{m^2}{10^n} - E \in [0, 1)$.

Starting with condition (ii), we have

$$\begin{aligned}
\frac{m^2}{10^n} - E &= \frac{d^2}{81}\cdot 10^n - \frac{2d^2}{81} + \frac{d^2}{81}\cdot 10^{-n} - \left(\frac{d^2}{81}\cdot 10^n - \frac{d^2}{81} - \frac{d}{9}\right)\\
&= \frac{d}{9} - \frac{d^2}{81} + \frac{d^2}{81}\cdot 10^{-n}\\
&= \frac{d}{9}\left(1-\frac{d}{9}\right) + \frac{d^2}{81}\cdot 10^{-n}\\
&\le \frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{10}\\
&= \frac{1}{4} + \frac{1}{10} < 1,
\end{aligned}$$

so condition (ii) always holds, since $n \ge 1$. So it remains to consider condition (i). This holds iff

$$d\left(d\frac{10^n-1}{9}-1\right) \equiv 0 \pmod 9$$

Reducing,

$$d(nd-1) \equiv 0 \pmod 9$$

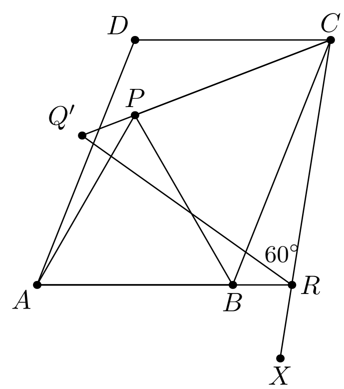
which holds iff either $d = 9$ or $nd \equiv 1 \pmod 9$. Hence, it remains to count pairs (n, d) with $1 \le n \le 2025$ and $1 \le d \le 9$ satisfying either $d = 9$ or $nd \equiv 1 \pmod 9$.

- For the $d = 9$ case, we have $n = 1$ to $n = 2025$, so 2025 possibilities.
- For the $nd \equiv 1 \pmod 9$ case, we can make n to be any value between 1 and 2025 relatively prime to 9, which is exactly $\frac{2}{3}$ of all such values (since 2025 is divisible by 9 and $\phi(9) = 6$). So we get $2025 \cdot \frac{2}{3} = 1350$ possibilities.

In total, we have $2025 + 1350 = 3375$ possibilities.

5. **Solution 1:** In the following solution, all angles are directed angles.

   Construct $\overrightarrow{CX}$ such that $\measuredangle PCX = 60^\circ$. If $\overline{XC}$ is parallel to $\overline{AB}$, then $\measuredangle PCX = \measuredangle PBA$ implies that C, P, B are collinear, so $\overleftrightarrow{CP} \parallel \overleftrightarrow{AD}$, which contradicts what we are given, namely that $\overleftrightarrow{CP}$ intersects $\overleftrightarrow{AD}$. Let therefore $\overleftrightarrow{XC}$ intersect $\overleftrightarrow{AB}$ at R. Let Q' be the unique point on $\overrightarrow{CP}$ such that $\measuredangle CRQ' = 60^\circ$; then $\triangle CRQ'$ is equilateral. We claim that $Q' = Q$, which is equivalent to showing that Q' lies on $\overline{AD}$, although we draw it as if Q' does not lie on $\overline{AD}$.



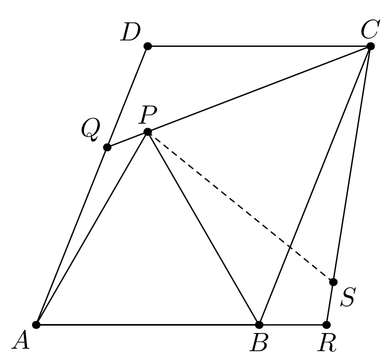
   Now $\measuredangle RQ'P = \measuredangle RQ'C = 60^\circ$, while $\measuredangle RAP = \measuredangle BAP = 60^\circ$. Hence R, Q', A, P are concyclic.

   At the same time, $\measuredangle PBR = 60^\circ = \measuredangle PCR$ (remember, we are using directed angles), so B, P, C, R are concyclic.

   It follows that $\measuredangle RBC = \measuredangle RPC = \measuredangle RPQ' = \measuredangle RAQ'$, so $\overline{AQ'} \parallel \overline{BC}$. Since the line through A parallel to $\overleftrightarrow{BC}$ can intersect line $\overleftrightarrow{CP}$ in a unique point (and both Q and Q' satisfy this property), we conclude that $Q = Q'$, so $\triangle CRQ$ is equilateral, and R is the desired point.

   Solution 2: In the following solution, all angles are directed angles.

   Construct equilateral triangle CPS (counterclockwise, so with S below line CP in the diagram below). As above, (using $X = S$), we find that $\overleftrightarrow{CS}$ cannot be parallel to $\overleftrightarrow{AB}$ by the information given in the problem statement, so suppose that $\overleftrightarrow{CS}$ intersects $\overleftrightarrow{AB}$ at a point R, shown below.



Then as R, A, and B are collinear, $\measuredangle RAP = \measuredangle BAP = 60°$, and as R, S, and C are collinear, $\measuredangle RSP = \measuredangle CSP = 60°$. Hence R, A, S, P are concyclic.

Rotation about P by 60 degrees counterclockwise turns A into B and S into C, and hence transforms $\triangle ASP$ into $\triangle BCP$. It follows that $\measuredangle PSA = \measuredangle PCB$, and since $\overline{BC} \parallel \overline{AD}$, $\measuredangle PCB = \measuredangle PQA$. Hence $\measuredangle PSA = \measuredangle PQA$, so Q is also on the circle through R, A, S, P.

Therefore $\measuredangle RQC = \measuredangle RQP = \measuredangle RAP = \measuredangle BAP = 60°$, so $\triangle RQC$ has two 60 degree angles and so is equilateral.

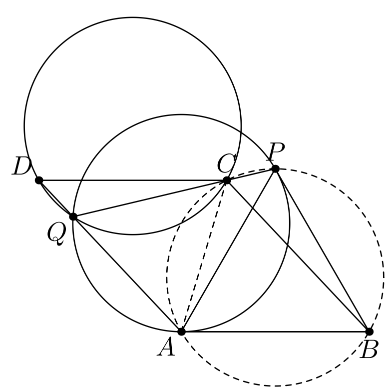
Solution 3: First, we will deal separately with the cases when A, P, and Q are collinear, as well as when C, D, and Q are collinear.

Note that if P lies on $\overleftrightarrow{AQ}$, then $Q = P$, and it follows that $\angle DAB = 60°$. Hence $\angle ABC = 120°$, so $\angle QBC = 120° - \angle PBA = 60°$. Since Q lies on $\overleftrightarrow{AD}$, we know that B, C, and Q are not collinear, so let $R \neq B$ be the second point where the circumcircle of $\triangle BCQ$ intersects $\overleftrightarrow{AB}$ (if it only intersects at B, then $\overline{AB}$ is tangent to the circle, so as $\angle QBA$ and $\angle QCB$ both inscribe arc QB, it follows that $\angle QCB = 60°$, so $R = B$ will work). Then $\measuredangle CRQ = \measuredangle CBQ = 60°$, and $\measuredangle RQC = \measuredangle RBC = 60°$, so it follows that $\triangle RQC$ is equilateral.

Next, if $A = Q$, then $\angle BAC = 60°$. Therefore, if R is the intersection of the circle of radius AC centered at A with $\overleftrightarrow{AB}$ such that $\angle RAC = 60°$, then we find that $\triangle CQR$ is equilateral. It follows that in all remaining cases, we may assume that A, Q, and P are noncollinear.

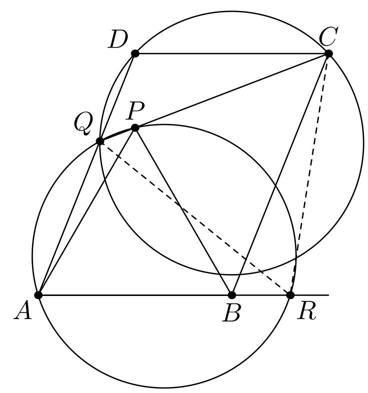
If Q lies on $\overleftrightarrow{CD}$, then $Q = D$, and P lies on D. Thus the distance between $\overline{AB}$ and $\overline{CD}$ is equal to the height of an equilateral triangle with base AB. It follows that if we construct an equilateral triangle with base $\overline{CD}$ toward the interior of the parallelogram (where we know $CD = AB$), then its third vertex R will lie on $\overline{AB}$. Since $Q = D$, this yields equilateral $\triangle CQR$. Otherwise, we may assume that C, D, and Q are noncollinear.

Thus we can construct the circumcircle O_1 of $\triangle APQ$ and O_2 of $\triangle CDQ$. If $\overleftrightarrow{AB}$ is tangent to O_1, then since $\measuredangle BAP = 60°$ and $\measuredangle AQP$ inscribes the same arc, then $\measuredangle AQP = 60°$.



But as $\overline{AQ} \parallel \overline{BC}$, $\measuredangle BCP = \measuredangle AQP = 60°$. Thus as $\measuredangle BAP = \measuredangle BCP = 60°$, then A, B, P, and C are concyclic. Therefore, $\measuredangle PCA = \measuredangle PBA = 60°$. It follows that $\measuredangle QCA = \measuredangle PCA = 60°$. Hence

$\triangle QAC$ has two 60° angles, so it is equilateral, and $R = A$ will work. Otherwise, let R be the second point of intersection of line $\overleftrightarrow{AB}$ with O_1. Then $\measuredangle PQR = \measuredangle PAB = 60^\circ$, since they both inscribe arc PR.



Let R_1 and R_2 be the radii of O_1 and O_2, respectively. Note that $\sin\angle AQP = \sin\angle CQD$, because the angles are supplementary. By the Extended Law of Sines,

$$2R_1 = \frac{AP}{\sin\angle AQP} \quad \text{and} \quad 2R_2 = \frac{CD}{\sin\angle CQD}.$$

Since $CD = AB = AP$, it follows that $R_1 = R_2$. Similarly, we note that $\angle QAB$ and $\angle CDA$ are supplementary, so $\sin\angle QAB = \sin\angle CDA$. Thus by the Extended Law of Sines,

$$QR = 2R_1 \sin\angle QAB \quad \text{and} \quad CQ = 2R_2 \sin\angle CDA.$$

It follows that $CQ = QR$. Triangle $\triangle CQR$ is therefore isosceles with one angle equal to 60°, and so it is equilateral, as desired.

Solution 4: We use complex numbers.

Align $\overline{AB}$ with the real axis in the complex plane so that B lies at the origin and without loss of generality, let $A = 1$. Draw the parallelogram so that $\overline{CD}$ lies above the real axis; since $\triangle ABP$ is equilateral, this means $P = \omega$, where $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a primitive sixth root of unity. Define $C = v \in \mathbb{C}$, so that $D = 1 + v$.

Consider the transformation of the complex plane that maps A to Q, B to C, and reflects $\triangle ABC$ (this can be thought of as a dilation/scaling, followed by a reflection, a rotation, and a translation). Note that these transformations all preserve angles. If we can show that this maps P to the real axis, then we are done, as the transformation preserves equilateral triangles, hence ABP maps to a triangle QCR with $R \in \mathbb{R}$. A reflection can be represented by taking the conjugate of z, while rotation/scaling can be represented by multiplying by a complex number; translation can be represented by adding a complex number.

Such a transformation can be written as $z \mapsto \alpha\overline{z} + v$, since it maps $B = 0$ to $C = v$. Since A maps to Q we have $Q = \alpha \cdot 1 + v = \alpha + v$. By construction, Q should lie on both lines AD and CP, hence

$$\frac{Q-D}{D-A} = \frac{(\alpha+v)-(1+v)}{(1+v)-1} \in \mathbb{R} \tag{1}$$

$$\frac{P-C}{Q-C} = \frac{\omega - v}{(\alpha+v)-v} \in \mathbb{R} \tag{2}$$

Simplifying (1), $\frac{\alpha-1}{v} \in \mathbb{R}$ so $v = \kappa(\alpha-1)$ for some $\kappa \in \mathbb{R}$. Simplifying (2), $\frac{\omega-v}{\alpha} \in \mathbb{R}$ so $\frac{\omega-\kappa\alpha+\kappa}{\alpha} \in \mathbb{R}$, hence

$$\frac{\omega+\kappa}{\alpha} \in \mathbb{R} \tag{3}$$

Finally, what we want to show is that P gets mapped to $\mathbb{R}$, i.e.

$$\alpha\overline{\omega} + v = \alpha\overline{\omega} + \kappa(\alpha-1) \in \mathbb{R}$$

This is true if and only if $\alpha(\overline{\omega}+\kappa) \in \mathbb{R}$; multiplying by (3) (which is a nonzero real number) we have

$$\alpha(\overline{\omega}+\kappa) \cdot \frac{\omega+\kappa}{\alpha} = (\overline{\omega}+\kappa)(\omega+\kappa),$$

which is a number times its complex conjugate, hence in $\mathbb{R}$ as desired.

Note: In fact, we didn't use the fact that ω is a sixth root of unity. Thus, this shows that any triangle ABP has a corresponding similar triangle with base CQ and with third point on line AB.

6. **Solution 1a:** Define a *friendly number pair* to be a pair of friendly numbers (r, s) with $r^r = s^s$. We will first show that all friendly number pairs (r, s) with $r < s$ have the form

$$(r,s) = \left(\left(\frac{q-1}{q}\right)^q, \left(\frac{q-1}{q}\right)^{q-1}\right)$$

for some integer $q \geq 2$. We then show that the former expression is strictly increasing and the latter strictly decreasing, both converging to $\frac{1}{e}$. Therefore, the second smallest friendly number is when $q = 3$:

$$\left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

For the first part, we first prove the following lemma:

Lemma: Let x, y, a, b be positive rational numbers with $x^a = y^b$. Then there exists a positive rational common base β and relatively prime positive integers m, n such that $x = \beta^m$ and $y = \beta^n$.

Proof. WLOG assume a, b are positive integers by taking the power to get get rid of the denominator in a and b. We may additionally assume WLOG that a, b are relatively prime by taking the root to divide out any common factor. Then, writing the prime factorization as $x = \prod_i p_i^{x_i}$ and $y = \prod_i p_i^{y_i}$ with $x_i, y_i \in \mathbb{Z}$, we have that $ax_i = by_i$ for all i. Since a and b are relatively prime it follows that there exists $n_i \in \mathbb{Z}$ such that $x_i = bn_i$ and $y_i = an_i$. Since this is true for all i, letting $\beta = \prod_i p_i^{n_i}$ we get $x = \beta^b$ and $y = \beta^a$, so the lemma is proven. □

Applying the lemma to the problem statement, we get $r = \beta^m$ and $s = \beta^n$ for some rational $\beta > 0$ and relatively prime $m, n \in \mathbb{N}$. Rewriting $r^r = s^s$, this gives

$$\beta^{m\beta^m} = \beta^{n\beta^n} \implies m\beta^m = n\beta^n.$$

Since $m, n > 0$,

$$\beta^{m-n} = \frac{n}{m}.$$

Since m and n are relatively prime and the above is fully reduced, this implies equality of the numerators and denominators; writing $\beta = \frac{p}{q}$ ($p, q > 0$ and $\gcd(p, q) = 1$) we get

$$n = p^k, m = q^k$$

where $k = m - n \in \mathbb{Z}$; in other words

$$k = m - n = q^k - p^k.$$

Possibly swapping r and s, we may take k to be positive, hence $q > p$ and we can factor

$$k = q^k - p^k = (q - p)(q^{k-1} + q^{k-2}p + \cdots + qp^{k-2} + p^{k-1}).$$

There are k terms in the second part of the RHS expression, and all terms are ≥ 1, so the only way equality can hold is if $q - p = 1$ and all terms are 1, hence $q^{k-1} = p^{k-1} = 1$, implying $k = 1$ since $p = q = 1$ violates $q > p$. Thus $\beta = \frac{q-1}{q}$, $m = q$, $n = p$, and getting back to our expressions for r and s,

$$r = \beta^m = \left(\frac{q-1}{q}\right)^q \quad \text{and} \quad s = \beta^n = \left(\frac{q-1}{q}\right)^{q-1}.$$

Noting that r is smaller than s above, this shows r and s always have the desired form when $r < s$. We check that any such r, s (for $q \geq 2$) satisfy the original equation $r^r = s^s$, so this completely describes the set of solutions.

To complete the problem, it remains to show that the second smallest is $\frac{8}{27}$. Letting $r(q), s(q)$ denote the expressions for r and s in terms of q (respectively), for $q \geq 2$, we claim that

$$r(q) < \frac{1}{e} < s(q)$$

with $r(q)$ strictly increasing and $s(q)$ strictly decreasing. This implies the second smallest is $r(3)$.

To show this we can use calculus. Taking the derivative:

$$\frac{d}{dq} \log\left(\frac{q-1}{q}\right)^{q-a} = \frac{d}{dq}(q-a)\left[\log(q-1) - \log q\right] = \log(q-1) - \log q + \frac{q-a}{(q-1)q}$$

from which we get

$$\frac{d}{dq}\log r(q) = \log(q-1) - \log q + \frac{1}{q-1}$$
$$\frac{d}{dq}\log s(q) = \log(q-1) - \log q + \frac{1}{q}.$$

Now taking the second derivative,

$$\begin{aligned}\frac{d^2}{dq^2}\log r(q) &= \frac{1}{q-1} - \frac{1}{q} - \frac{1}{(q-1)^2} \\ &= \frac{1}{(q-1)q} - \frac{1}{(q-1)^2} \\ &= \frac{-1}{(q-1)^2 q} \\ \frac{d^2}{dq^2}\log s(q) &= \frac{1}{q-1} - \frac{1}{q} - \frac{1}{q^2} \\ &= \frac{1}{(q-1)q} - \frac{1}{q^2} \\ &= \frac{1}{(q-1)q^2}.\end{aligned}$$

The former is strictly negative and the latter strictly positive, therefore $r(q)$ is concave down and $s(q)$ concave up. Moreover, both $r(q)$ and $s(q)$ approach $\frac{1}{e}$ as $q \to \infty$, since

$$\left(\frac{q-1}{q}\right)^q = \left(1 - \frac{1}{q}\right)^q \to e^{-1} = \frac{1}{e}.$$

A concave down function can only converge as $q \to \infty$ if it is decreasing, and a concave up function only if it is increasing. Thus, the remaining claim is proven.

Solution 1b: We follow the approach above, but provide an alternate proof that $r(n) < r(n+1)$ and $s(n+1) < s(n)$ for all integers $n \geq 2$, where, as above,

$$r(n) = \left(\frac{n-1}{n}\right)^n \quad \text{and} \quad s(n) = \left(\frac{n-1}{n}\right)^{n-1}.$$

Applying AM-GM to n numbers $1 - 1/n$ and 1, we get

$$(1 - 1/n)^{n/(n+1)} \leq \frac{n(1-1/n)+1}{n+1} = 1 - \frac{1}{n+1}.$$

Raising this to the $n+1$ power, we find $r(n) < r(n+1)$.

On the other hand, applying AM-GM with $(n-1)$ numbers $1+1/(n-1)$ and 1, we get

$$(1+1/(n-1))^{(n-1)/n} \le \frac{(n-1)(1+1/(n-1))+1}{n} = \frac{n+1}{n}.$$

This can be written as $\left(\frac{n}{n-1}\right)^{n-1} < \left(\frac{n+1}{n}\right)^n$, or rather $\left(\frac{n}{n+1}\right)^n < \left(\frac{n-1}{n}\right)^{n-1}$. This can be written as $\left(1-\frac{1}{n+1}\right)^n < \left(1-\frac{1}{n}\right)^{n-1}$. Thereore $s(n+1) < s(n)$. This shows the same result as above, without calculus.

Solution 2: Suppose $r^r = s^s$ and assume without loss of generality that $r > s$. If $x, y > 1$, then x^y is increasing in both arguments, so x^x is increasing for $x \ge 1$, and clearly $x^x \ge 1$ on this range. On the other hand, if $x < 1$, then $x^x < 1$. All friendly numbers must therefore be between 0 and 1 (exclusive).

The equation $r^r = s^s$ implies that $r^{r/s} = s$; let $u = r/s$, so $s = r^u$ and $u > 1$. Again, $r^r = s^s$ implies $r^r = r^{ur^u}$. Since $r \ne 1$, it follows that $r = ur^u$, i.e. $u = r^{1-u}$. (In fact, since $s = r/u$, the set of friendly numbers corresponds to the set of rational numbers u such that $u^{1/(1-u)}$ is rational.)

We know $u > 1$, so let $v = u - 1$; then v is a positive rational number. Let also $q = r^{-1}$, so q is a rational number greater than 1. Then $q^v = 1 + v$.

Suppose that $v = m/n$ with m, n relatively prime. Then $q^{m/n} = \frac{m+n}{n}$, so $\frac{m+n}{n}$ is a perfect m^{th} power of a rational. Clearly $\gcd(m+n, n) = \gcd(m, n) = 1$, so it follows that $m+n$ and n are both perfect m^{th} powers of integers, i.e. $n = j^m$, $m+n = k^m$ for positive integers $j < k$. Assume now that $m \ge 2$. The function $(x+1)^m - x^m$ is a polynomial of positive degree $m-1$ with only positive coefficients, so it is increasing on $x \ge 0$. Hence for $x \ge 1$, $(x+1)^m - x^m \ge 2^m - 1$. But then $(m+n) - n = m = k^m - j^m \ge 2^m - 1$, and this is impossible because $2^m > m+1$ for $m \ge 2$ (by induction: it holds when $m = 2$, and if $2^m > m+1$, then $2^{m+1} > 2m+2 > m+2$).

The only possibility remaining is $m = 1$, where $v = 1/n$, $q^{1/n} = 1 + \frac{1}{n}$, so $q = (1+\frac{1}{n})^n$. Hence $r = 1/q = (\frac{n}{n+1})^n$, $u = v + 1 = \frac{n+1}{n}$, and $s = r^u = (\frac{n}{n+1})^{n+1}$. The algebra above implies that $r^r = s^s$, and in fact

$$r^r = \left(\frac{n}{n+1}\right)^{rn} = \left(\frac{n}{n+1}\right)^{\frac{n^{n+1}}{(n+1)^n}} = \left(\frac{n}{n+1}\right)^{s(n+1)} = s^s.$$

All friendly numbers are thus contained in this countably infinite family of pairs r_n, s_n. We claim

$$s_1 < s_2 < \cdots < s_n < \cdots < r_n < \cdots < r_2 < r_1,$$

and since we already know that $s_n < r_n$, the ordering will follow upon showing that the s_n increase and the r_n decrease.

Here it is possible to use calculus (see Solution 1), and the fact that the r_n decrease is also an immediate consequence of the well-known statement that $(1+\frac{1}{n})^n$ increases to e (in fact, the unique real number that is smaller than all the r_n and larger than all the s_n is $1/e$). Here we give another way to prove the assertions without any calculus.

We wish to prove that $(1+\frac{1}{n})^n$ is increasing and (after replacing n with $n-1$) that $(1-\frac{1}{n})^n$ is also increasing. The binomial expansion gives

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1-\frac{j}{n}\right) = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=1}^{k-1} \left(1-\frac{j}{n}\right).$$

For fixed j, the term $1-\frac{j}{n}$ is increasing in n, and the series for larger n also has more (nonnegative) terms. Hence $(1+\frac{1}{n})^n$ is increasing in n.

A similar argument works with $(1-\frac{1}{n})^n$, but it is necessary to be more careful due to the fact that the terms in the binomial expansion alternate in sign. Let $2g+1$ be the smallest odd integer greater than or equal to n. We have

$$\left(1-\frac{1}{n}\right)^n = \sum_{k=0}^{2g+1} \binom{n}{k} \frac{(-1)^k}{n^k} = \sum_{i=0}^{g} \left[\binom{n}{2i}\frac{1}{n^{2i}} - \binom{n}{2i+1}\frac{1}{n^{2i+1}}\right] =$$

$$\sum_{i=0}^{g} \left[\frac{1}{2i!} \prod_{j=0}^{2i-1} \left(1-\frac{j}{n}\right) - \frac{1}{(2i+1)!} \prod_{j=0}^{2i} \left(1-\frac{j}{n}\right)\right] =$$

$$\sum_{i=0}^{g} \frac{1}{(2i)!} \left[1-\frac{1-2i/n}{2i+1}\right] \prod_{j=0}^{2i} \left(1-\frac{j}{n}\right) = \sum_{i=0}^{g} \frac{1}{(2i)!} \frac{2i}{2i+1} \left[1+\frac{1}{n}\right] \prod_{j=1}^{2i} \left(1-\frac{j}{n}\right) =$$

$$\sum_{i=0}^{g} \frac{1}{2i!} \frac{2i}{2i+1} \left(1-\frac{1}{n^2}\right) \prod_{j=2}^{2i} \left(1-\frac{j}{n}\right).$$

Here again all terms involving n are increasing in n, and as n increases there may also be more nonnegative terms. We have thus established that $(1-\frac{1}{n})^n$ is increasing in n.

It follows that the second smallest friendly number is $s_2 = (\frac{2}{3})^3 = \frac{8}{27}$.

Note: This problem was similar to a problem that appeared on the USA Mathematical Talent Search (USAMTS), Year 32 (2020-2021), Round 1, Problem 5.[1] The problem writing committee regrets the overlap.

[1] `https://files.usamts.org/Problems_32_1.pdf`