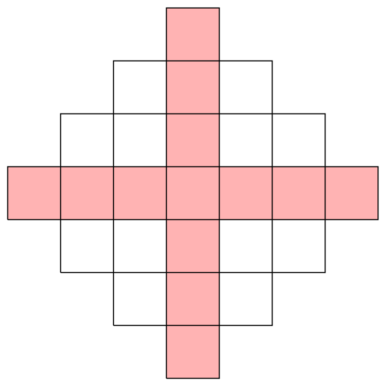
2023 Utah Math Olympiad Solutions March 2023

1. Scarlett will win if she uses the following strategy:

   - On the first turn, Scarlett selects the center square.
   - On each subsequent turn, if Indigo selects square X in her turn, then Scarlett selects the square that is a 180° rotation about the center square.

   We claim that if Scarlett follows this strategy, then she will always be able to make a move on her turn. After Scarlett's first turn, the board looks like this:

   Notice that the board has a 180° symmetry: if a square is painted, then its 180° rotation is also painted, and vice versa. Additionally, the only squares X whose 180° rotations lie in the same row or column are the squares X in the center row or center column, which are already painted. Since Indigo cannot play in any of these squares, her move will necessarily allow Scarlett to select the 180° rotation of X on her turn.

   This means that after Scarlett's turn, the symmetry of the board is preserved, since the squares that she paints are exactly the 180° rotations of the squares that Indigo painted. Therefore, Scarlett's strategy continues to work after each of Indigo's turns. Since only a finite number of moves can be made, it follows that the game must end with Indigo unable to make a move, and Scarlett wins.

2. The answer is $N = 2$ and the possibilities are $A = 33$ and $A = 34$.

   Note that $A/99$ can be written as a repeating decimal with two-digit blocks, each of which are equal to A. For instance, $37/99 = 0.373737\ldots$. However, $99/99 = 1.000\ldots$. Thus, if Abigail tells Chris a first digit C after the decimal point of $A/99$, then Chris will be able to list 10 possibilities for A unless $C = 9$, in which case Chris can only list the nine possibilities $A = 90, 91, \ldots, 98$. Therefore, when Bruno is told the first digit B of $A/11$, he must know that A is not any of $90, 91, \ldots, 98$. Since $\frac{A+11}{11} = \frac{A}{11} + 1$ has the same first digit after the decimal point as $\frac{A}{11}$, it follows that the digit B only depends on the value of A modulo 11. In particular, if $A \equiv 0, 1 \pmod{11}$, then $B = 0$, while if $A \equiv k \pmod{11}$ for $2 \leq k \leq 10$, then $B = k - 1$. Note that $A = 90, 91, \ldots, 98$ give $B = 2, 3, \ldots, 10$, respectively, which all must be impossible as otherwise Bruno could not deduce that Chris can narrow A down to 10 possibilities. Therefore, $A \equiv 0$ or $A \equiv 1$ mod 11, and $B = 0$.

   Following the logic in the above paragraph, Chris deduces that $B = 0$, and thus that $A \equiv 0$ or $A \equiv 1 \pmod{11}$. For each possible value of C, we now list possible values of A which are 0 or 1 mod 11:

C	Possible values for A
0	1, 99
1	11, 12
2	22, 23
3	33, 34
4	44, 45
5	55, 56
6	66, 67
7	77, 78
8	88, 89

It remains to check for which of these cases all values for A are the product of two distinct primes. The only row satisfying this criteria is $C = 3$, with $A = 33 = 3 \cdot 11$ or $A = 34 = 2 \cdot 17$. Therefore, $C = 3$, $N = 2$, and either $A = 33$ or $A = 34$.

3. We claim the answer is $90°$. This is achievable when $PQRS$ is a square.

   To show that this is minimal, let C be the convex closure of P, Q, R, S. (The *convex closure* of a set of points is the smallest convex polygon containing those points.) If C is a quadrilateral, then its internal angles add up to $360°$, so there must be at least one angle greater than or equal to $90°$. If any three points are collinear, then the angle between those three points is $180°$. Otherwise, without loss of generality, say S lies inside triangle PQR. Then angles PSQ, QSR, and RSP add up to $360°$, and are all less than $180°$. So one of them is at least $120°$.

4. There are 83 possibilities for Daniel's function. We consider the following three cases:

   Case 1: $m = 0$. If Daniel's function is the last function to be pulled out of the hat, the final answer will be $0 \cdot x + b = b$. Therefore, we must have $b = 1$. Additionally, $m = 0$ and $b = 1$ is achievable, because if everyone's function is the constant function $f(x) = 1$, the end result will always be 1. So there is 1 possibility in this case.

   Case 2: $m \geq 1$ and $b = 0$. Suppose that the other nine functions are $g_1, g_2, \ldots, g_9$. If we apply the functions in the order $f, g_9, g_8, \ldots, g_1$, then since $f(0) = m \cdot 0 = 0$, we obtain

   $$g_1(g_2(\cdots g_9(f(0))\cdots)) = g_1(g_2(\cdots g_9(0)\cdots)) = 1.$$

   But if we apply the functions in the order $g_9, g_8, \cdots g_1, f$, then we obtain

   $$f(g_1(g_2(\cdots g_9(0)\cdots))) = f(1) = m \cdot 1 = m.$$

   Therefore we must have $m = 1$. Additionally, $m = 1$ and $b = 0$ is achievable by letting $f(x) = x$ and $g_i(x)$ be the constant function 1 for all i. So there is also 1 possibility in this case.

   Case 3: $m, b \geq 1$. We claim that we can pick nine other functions such that if the functions are applied in any order, then the end result is always 1. We will define everyone's function to be the same function g, which will be based off of Daniel's favorite function. To show how we arrive at

g, first imagine that everyone's favorite function was Daniel's favorite function, that is, $g = f$. Then we get:

$$\begin{aligned} f(0) &= b \\ f^2(0) &= bm + b \\ f^3(0) &= bm^2 + bm + b \\ &\vdots \\ f^{10}(0) &= bm^9 + bm^8 + \cdots + bm + b. \end{aligned}$$

Note that this is a strictly increasing sequence of positive integers. But $f^{10}(0)$ does not take us back to 1. So rather than using $g = f$, we use the slightly modified

$$g(x) = \begin{cases} \frac{1-b}{m} & \text{if } x = f^8(0) \\ 1 & \text{if } x = f^9(0) \\ f(x) & \text{otherwise.} \end{cases}$$

Since f and g do the same things to 0, $f(0)$, $f^2(0)$, $\ldots$, and $f^7(0)$ (and since all of these are distinct numbers), we can assume that after 8 functions have been pulled out of the hat, 0 has been turned into $f^8(0)$. After this, we find:

- If Daniel's function is picked next, then the next value is $f^9(0)$, and then $g(x)$ is picked, last, so the final value is $g(f^9(0)) = 1$.
- If g is picked next, then the next value is $g(f^8(0)) = \frac{1-b}{m}$, so the final value is $f\left(\frac{1-b}{m}\right) = m \cdot \frac{1-b}{m} + b = 1$.

In either case, the final value is 1, so if $m, b \geq 1$, then we have shown that $f(x) = mx + b$ could possibly be Daniel's function. Therefore, this case has $9 \cdot 9 = 81$ possibilities.

Note: There are other choices for g that work; for example, we can take

$$g(x) = \begin{cases} \frac{1-b}{m} & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

In summary, we have demonstrated that (m, b) can be $(0, 1)$, $(1, 0)$, or we can have $m, b \geq 1$, for a total of $1 + 1 + 9^2 = 83$ functions.

5. The optimal strategy is to roll one die and keep it only if it is a 6, and if not to roll the other 2022 dice and keep the maximum of those. Call this strategy $(*)$.

First, Noureddine should only divide the dice into two piles; if he divides them into three piles with A, B, and C dice, then he can do at least as well with piles of $A + B$ and C dice by making the same decision (keep or discard) after seeing the $A + B$ dice that he would by seeing the A and B dice separately, and maybe he can do better by seeing more information at once. The same argument applies to any number of piles greater than 3, so we may assume there are only two piles.

Let the two piles have sizes A and B, where $A + B = 2023$. Let $E(N)$ be the expected value of the maximum of N dice. For example, $E(1) = 3.5$. If A is fixed, then Noureddine's best strategy is to roll the first pile, keep the maximum if it's greater than $E(B)$, and otherwise roll and keep the maximum of the second pile. Additionally, we expect that $E(N)$ should quickly approach 6 as N gets large, as it is very likely to roll at least one 6. Concretely, $E(N)$ is given by the formula

$$E(N) = 6 - \frac{5^N + 4^N + 3^N + 2^N + 1}{6^N},$$

where this formula computes the maximum by subtracting one at a time: $\left(\frac{5}{6}\right)^N$ is the probability that all dice are at most 5, $\left(\frac{4}{6}\right)^N$ is the probability that all dice are at most 4, and so on. Observe that $(5/6)^N$, $(4/6)^N$, ..., $(1/6)^N$ are all decreasing functions of N, so when they are subtracted from 6, it follows that $E(N)$ is an increasing function of N. Using this formula, the first few values of $E(N)$ are

$$\begin{aligned} E(1) &= 6 - \frac{15}{6} &&= 3 + \frac{1}{2} \\ E(2) &= 6 - \frac{55}{36} &&= 4 + \frac{17}{36} \\ E(3) &= 6 - \frac{225}{216} &&= 4 + \frac{207}{216} \\ E(4) &= 6 - \frac{979}{1296} &&= 5 + \frac{317}{1296}. \end{aligned}$$

We first consider the case $B \geq 4$, i.e., $A \leq 2019$. Since $E(B) \geq 5$, Noureddine keeps one of the A dice only if he rolls a 6. Comparing this strategy to $(*)$, we notice that both strategies score 6 if any of the dice are a 6. However, *given* that all the dice are 5 or fewer, Noureddine's expected score for $(*)$ is $E(2022)$, whereas his expected score for $A \geq 2$ is $E(2023 - A) < E(2022)$. Therefore, this case is not optimal unless $A = 1$.

It then remains to consider the case where $B < 4$, i.e. $A \geq 2020$. In this case, $E(B) < 5$. We claim that this case is not optimal either, by again comparing to what happens if strategy $(*)$ is applied to the same dice.

- If there is a 6 among the A dice, then both strategies score 6.
- If there is a 6 among the B dice but not among the A dice, and the maximum of the A dice is exactly 5, then strategy $(*)$ scores at least 1 higher. This happens with probability

$$\left(\left(\frac{5}{6}\right)^A - \left(\frac{4}{6}\right)^A\right)\left(1 - \left(\frac{5}{6}\right)^B\right) > \left(\frac{5^A - 4^A}{6^A}\right) \cdot \frac{1}{6} \quad \text{since } B \geq 1.$$

- Finally, if there are no 6s, then strategy $(*)$ might score better, the same, or worse. But it only scores worse if the first die is less than 6, strictly larger than all other dice, and good enough to keep (above $f(B)$, which is at least 3.5). So the first die must be a 4 or 5, and the last 2022 dice must be $1, 2, 3$, or 4. This happens with probability at most $\frac{2}{6}\left(\frac{4}{6}\right)^{2022}$, and the amount that it scores worse is at most 4.

In total, the expected advantage of strategy $(*)$ over the candidate strategy is *at least*

$$\left(\frac{5^A - 4^A}{6^A}\right) \cdot \frac{1}{6} - 4 \cdot \frac{2}{6} \cdot \left(\frac{4}{6}\right)^{2022} = \frac{5^A 6^B - 4^A 6^B - 12 \cdot 4^{2023}}{6^{2024}}.$$

Since $4^{2023} = 4^{A+B} < 4^A 6^B$, dividing out by 6^B the numerator is at least $5^A - 13 \cdot 4^A$, which is positive since $A \geq 16$. Specifically, $\left(\frac{5}{4}\right)^4 = \frac{625}{256} > 2$, so $\left(\frac{5}{4}\right)^A \geq \left(\frac{5}{4}\right)^{4 \cdot 4} > 2^4 = 16 > 13$.

In summary, we have shown that it cannot be optimal to divide into two piles A, B (with $B < 4$), nor can it be optimal to divide into two piles A, B with $B \geq 4$ and $A \geq 2$. Therefore, the only remaining case is $A = 1$ and $B = 2022$, so strategy $(*)$ is optimal.

6. We prove that there are only four such operations $x \mid y$: $\max(x, y)$, $\min(x, y)$, $\text{first}(x, y) = x$, and $\text{last}(x, y) = y$.

   In the following lemmas, we first show idempotence $x \mid x = x$, with some work. Then we look at chains of sums of $(0 \mid 1)$ to find $0 \mid n = n(0 \mid 1)$, and finally we determine that $(0 \mid 1)$ is either 0 or 1. Combining this with the symmetric observation that $(1 \mid 0)$ is either 0 or 1 will give us the four cases.

   Lemma 1: $x \mid x = x$.

   Since $x \mid x = x + (0 \mid 0)$, it suffices to show $0 \mid 0 = 0$. Consider the map $\phi : \mathbb{Z}^+ \to \mathbb{Z}$ (where $\mathbb{Z}$ is the integers and $\mathbb{Z}^+$ is the positive integers) defined by $\phi(n) = \underbrace{0 \mid 0 \mid \cdots \mid 0}_{n \text{ zeros}}$. Then we claim

$$\phi(mn) = \phi(m) + \phi(n).$$

   This is by rewriting the right-hand side as

$$\underbrace{0 \mid 0 \mid \cdots \mid 0}_{m} + \phi(n) = \underbrace{\phi(n) \mid \cdots \mid \phi(n)}_{m}$$

   and expanding. In particular, we have that $\phi(p^a) = a\phi(p)$ for any prime p.

   Now we claim that $\phi(x) = \phi(y)$ for some $0 \leq x < y$. If $\phi(p) = 0$ for any prime, then $\phi(1) = \phi(p)$. Otherwise, there must be primes p_1 and p_2 such that $\phi(p_1)$ and $\phi(p_2)$ have the same sign. Examining $\phi(p_1^a) = a\phi(p_1)$ and $\phi(p_2^b) = b\phi(p_2)$, we see that we can choose $a, b > 0$ to make these equal: in particular, $a = |\phi(p_2)|$ and $b = |\phi(p_1)|$.

   To complete the proof of Lemma 1, $\phi(x) = \phi(y)$ implies that $\phi(n)$ is periodic for n sufficiently large, since $\phi(n) = \phi(n - x) \mid \phi(x) = \phi(n - x) \mid \phi(y) = \phi(n - x + y)$, and hence bounded. But $\phi(2^a) = a\phi(2)$ is not bounded unless $\phi(2) = 0$, so $\phi(2) = 0 \mid 0 = 0$, and in fact, $\phi(n) = 0$ for all n.

   Lemma 2: For $n \geq 0$, $0 \mid 1 \mid \cdots \mid n = n(0 \mid 1)$.

   Proof by induction:

$$\begin{aligned} 0 \mid 1 \mid 2 \mid \cdots \mid n \mid (n+1) &= 0 \mid (1 \mid 1) \mid (2 \mid 2) \mid \cdots \mid (n \mid n) \mid (n+1) \quad \text{by Lemma 1} \\ &= (0 \mid 1) \mid (1 \mid 2) \mid \cdots \mid (n \mid (n+1)) \\ &= (0 + (0 \mid 1)) \mid (1 + (0 \mid 1)) \mid (2 + (0 \mid 1)) + \cdots \\ &= (0 \mid 1 \mid \cdots \mid n) + (0 \mid 1). \end{aligned}$$

Lemma 3: For $n \geq 0$, $0 \mid n = n(0 \mid 1)$.

Note that:

$$\begin{aligned}(0 \mid n) + (0 \mid 1 \mid 2 \mid ... \mid n) &= (0 + (0 \mid 1 \mid 2 \mid ... \mid n)) \mid (n + (0 \mid 1 \mid 2 \mid ... \mid n)) \\ &= 0 \mid 1 \mid 2 \mid ... \mid (2n) \quad \text{by Lemma 1: } n \mid n = n\end{aligned}$$

Applying Lemma 2, $(0 \mid n) + n(0 \mid 1) = 2n(0 \mid 1)$, and the result follows.

Lemma 4: $(0 \mid 1) \in \{0, 1\}$.

Let $k = (0 \mid 1)$. By idempotence, $0 \mid 1 = 0 \mid 0 \mid 1 \mid 1$, and we consider different ways to evaluate this associatively. First, $(0 \mid 1) \mid 1 = k \mid 1$, and second, $0 \mid (0 \mid 1) = 0 \mid k$. Thus,

$$k = k \mid 1 = 0 \mid k$$

Now we have two cases. If $k \geq 0$, then

$$0 \mid k = k(0 \mid 1) = k^2,$$

so $k = k^2$ and $k \in \{0, 1\}$. Second, if $k < 0$, then subtracting k from $k = k \mid 1$ we get

$$0 = (k - k) \mid (1 - k) = 0 \mid (1 - k) = (1 - k)(0 \mid 1) = (1 - k)k$$

so again $k = 0$ or $k = 1$ (actually a contradiction since $k < 0$), and we are done.

Putting things together: All of lemmas 2-4 can be proven identically for the symmetric case of $b \mid a$ instead of $a \mid b$, from which we get that $1 \mid 0 \in \{0, 1\}$. So there are two cases for $0 \mid 1$ and two cases for $1 \mid 0$. Together with Lemma 3 we can then calculate $m \mid n$ for any m, n:

$$m \mid n = \begin{cases} m + (0 \mid (n - m)) = m + (n - m)(0 \mid 1) \text{ if } n \geq m \\ n + ((m - n) \mid 0) = n + (m - n)(1 \mid 0) \text{ if } m \geq n. \end{cases}$$

In particular:

- If $0 \mid 1 = 1 \mid 0 = 1$, this gives the max operation $\max(x, y)$.
- If $0 \mid 1 = 1 \mid 0 = 0$, this gives the min operation $\min(x, y)$.
- If $0 \mid 1 = 0$ and $1 \mid 0 = 1$, this gives the function first$(x, y) = x$.
- Finally, if $0 \mid 1 = 1$ and $1 \mid 0 = 0$, this gives the function last$(x, y) = y$.

It's easy to verify that each of these operations satisfy the given two properties, so these are the only four possible binary operations.