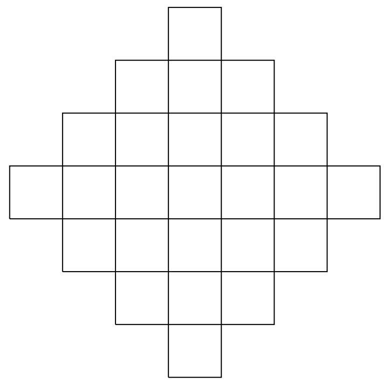
2023 Utah Math Olympiad March 25, 2023

1. Scarlett and Indigo take turns painting squares on the following board, with Scarlett going first. On their turn, a player selects an unpainted square and paints all the squares in the row and column containing that square (painting over painted squares if necessary).



   A player loses if they cannot make a move on their turn. Find, with proof, which player has a winning strategy.

2. Abigail is thinking of an integer A between 1 and 99, inclusive. She tells Bruno the first digit after the decimal point of the number $A/11$ (where this digit is 0 if $A/11$ is an integer). Then she tells Chris the first digit after the decimal point of the number $A/99$. Bruno says "I don't know what A is, but I know that Chris can narrow it down to a list of exactly 10 possibilities." Chris replies "Aha! Now I know that A is one of a list of N possibilities, all of which are a product of exactly two distinct primes."

   What is N, and what are Chris's N possibilities for Abigail's number A?

3. For a given choice of distinct points P, Q, R, and S in the plane, let $M(P, Q, R, S)$ be the maximum angle between any three of the four points. Find the minimum possible value of $M(P, Q, R, S)$, and demonstrate an arrangement of P, Q, R, S for which this is achieved.

4. Daniel goes to a party with nine other attendees. At the party, every attendee writes their favorite function from the real numbers to the real numbers on a piece of paper and puts it into a hat.

   Remarkably, it turns out that, if the ten functions are pulled out of the hat in any order and successively applied to the number 0, the end result is always 1. In other words, if the functions are pulled out in the order $f_1, f_2, f_3, \ldots, f_{10}$, then
$$f_{10}(f_9(\cdots(f_2(f_1(0)))\cdots)) = 1.$$

   If Daniel's favorite function is of the form $f(x) = mx + b$, where m and b are single-digit integers (from 0 to 9 inclusive), how many possibilities are there for Daniel's favorite function?

5. Noureddine is playing a game with 2023 fair six-sided dice. At the beginning of the game, he must divide the dice into two or more nonempty piles. Then he rolls all the dice in the first pile. He may either keep one die in the first pile and stop rolling, or discard the first pile and roll all the dice in the second pile. If he chooses to roll, he may either keep one die in the second pile and stop rolling, or discard the second pile and roll all the dice in the third pile, and so on. If Noureddine gets to the last pile, he must keep one of the dice in that pile. Noureddine's final score is the value of the die he keeps.

   Describe the strategy Noureddine should follow to maximize his expected score, and prove that it is optimal.

6. Find, with proof, all binary operations $\mid$ on the integers such that for all integers a, b, and c:
$$(a \mid b) + c = (a + c) \mid (b + c)$$
$$(a \mid b) \mid c = a \mid (b \mid c).$$

   Note: A binary operation $\mid$ is a function that for any two integers x and y, returns an integer $x \mid y$. For example, addition $x + y$ and multiplication $x \times y$ are binary operations.