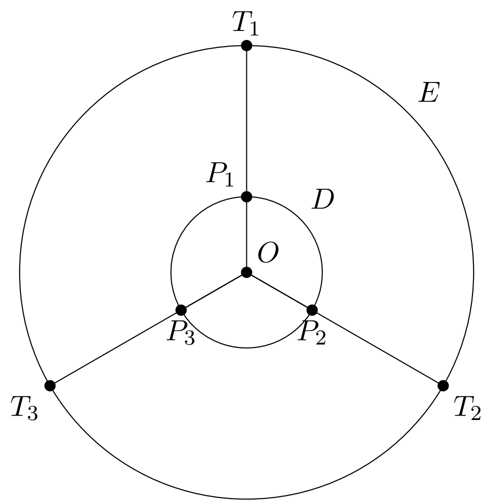
1. If x is even then $x^2 + bx + c \equiv 0 + 0 + 1 = 1 \bmod 2$, so it cannot be a root. If x is odd then $x^2 + bx + c \equiv 1 + b + 1 \equiv b \bmod 2$, so it cannot be a root unless b is even. Therefore in order for the polynomial to have an integer root, b must be an even prime and thus equal to 2.

   Next we write $x^2 + 2x + c = (x+1)^2 + (c-1)$, which is never zero for $c \geq 2$, so $c = 1$ and $x = -1$. Thus the only solution is $(x, b, c) = (-1, 2, 1)$.

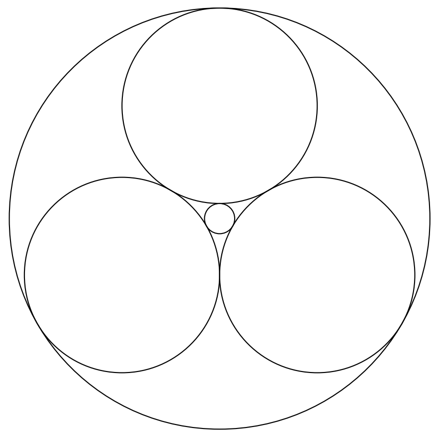
2. The three circles C_1, C_2, and C_3 must have the same radius.

   Suppose that we only draw circles D and E, marking the points where C_1, C_2, and C_3 are tangent to E as T_1, T_2, and T_3, respectively. We also draw radii from the common center of D and E (call it O) to T_1, T_2, and T_3. Suppose that the radii intersect circle D at P_1, P_2, and P_3, respectively.

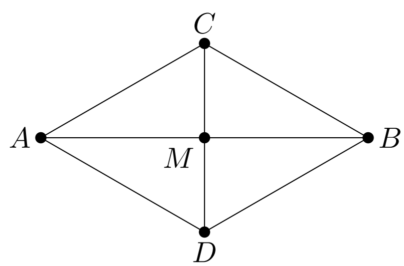


   If we were to draw the tangent lines to circle E at T_1, T_2, and T_3, then $\overline{OT_1}$, $\overline{OT_2}$, and $\overline{OT_3}$ would be perpendicular to the tangent line. The same applies for the radii of circles C_1, C_2 and C_3 drawn to T_1, T_2, and T_3, respectively. Hence the centers of C_1, C_2, and C_3 must lie on $\overline{OT_1}$, $\overline{OT_2}$, and $\overline{OT_3}$, respectively. Since the point of tangency between C_1 and D must lie on the line connecting the centers of the two circles, we see that P_1 must be the point of tangency between the two circles. It follows that $\overline{P_1T_1}$ is a diameter of circle C_1, and similarly, $\overline{P_2T_2}$ and $\overline{P_3T_3}$ are diameters of C_2 and C_3, respectively. If the radius of E is R and the radius of D is r, then $P_1T_1 = P_2T_2 = P_3T_3 = R - r$, so the three circles must have the same diameter, which implies that they must have the same radius.

   Note: Our diagram is not to scale—it turns out that circle D needs to be smaller. In fact, if the radius of circle D is 1, then C_1, C_2, and C_3 must have radius $3 + 2\sqrt{3}$, so E has radius $7 + 4\sqrt{3}$. The following picture shows a to-scale diagram.



3. **Solution 1:** It is impossible to assign a number to each point in the plane such that the property is true. Given *any* two distinct points A and B in the plane, let M be the midpoint of $\overline{AB}$. We can construct points C and D on the perpendicular bisector of $\overline{AB}$, such that $CM = MD = \frac{AM}{\sqrt{3}}$. Note that $\triangle ACD$ and $\triangle BCD$ are congruent equilateral triangles.

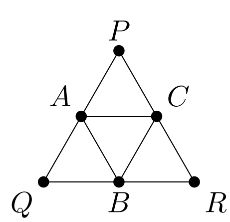


If such a function exists, then as $\triangle ACD$ and $\triangle BCD$ have the same perimeter, we know that

$$f(A) + f(C) + f(D) = f(B) + f(C) + f(D).$$

Hence $f(A) = f(B)$ for all distinct points A and B, so f must be equal to a constant for all points in the plane. If $f(P) = k$ for all points P in the plane, then it follows that $f(P) + f(Q) + f(R) = 3k$ for every equilateral triangle PQR. Hence the perimeter of every equilateral triangle in the plane is $3k$, an obvious contradiction. We conclude that the requested task is impossible.

Solution 2: Suppose for sake of contradiction that it is possible to make such an assignment. Let PQR be an equilateral triangle with side length 2, and let A, B, and C be the midpoints of $\overline{PQ}$, $\overline{QR}$, and $\overline{RP}$, respectively.



Note that $\triangle PAC$, $\triangle QBA$, $\triangle RCB$ and $\triangle ABC$ are equilateral triangles with side length 1. Then $f(P) + f(Q) + f(R) = 6$ and

$$\begin{aligned}(f(P) + f(A) + f(C)) &= 3\\ (f(Q) + f(B) + f(A)) &= 3\\ (f(R) + f(C) + f(B)) &= 3\\ (f(A) + f(B) + f(C)) &= 3\end{aligned}$$

Adding these equations, and subtracting $f(P) + f(Q) + f(R) = 6$, we find

$$3\,(f(A) + f(B) + f(C)) = 6.$$

which implies that $f(A) + f(B) + f(C) = 2$. But the perimeter of $\triangle ABC$ is 3, so

$$f(A) + f(B) + f(C) = 3,$$

and we have a contradiction. Thus no such assignment exists.

4. **Solution 1:** The answer is that $P_c(2021)$ is a multiple of 3 if and only if $c \equiv 2 \bmod 3$.

   First, scale the problem so that cows only have two legs and ostriches have one, and the number of animals on day n is equal to the number of legs on day $n-1$. Then, the number of animals Georgia adds on day n is simply the number of cows on day $n-1$, since they have one excess leg.

   The possible numbers of cows on day 2 are $c, c+1, c+2, \ldots, 2c$. Thus, we have the recursion

$$P_c(n) = \sum_{i=c}^{2c} P_i(n-1)$$

   for $n \geq 2$, with the initial condition $P_c(1) = 1$ for all c.

   We now claim that for all $n \geq 2$ we have $P_c(n) \equiv c+1 \pmod 3$. We prove this by induction. The base case is day 2, on which Georgia can add $0, 1, 2, \ldots c$ cows, which gives $c+1$ possibilities. Now suppose that $P_c(n) \equiv c+1 \pmod 3$ for all c; we show that $P_c(n+1) \equiv c+1 \mod 3$ as well.

   Working mod 3, we have

$$\begin{aligned}P_c(n+1) &\equiv \sum_{i=c}^{2c}(i+1)\\ &\equiv c+1+\sum_{i=c}^{2c} i\\ &\equiv c+1+\frac{2c(2c+1)}{2}-\frac{c(c-1)}{2}\\ &\equiv c+1+\frac{3c^2+3c}{2}\\ &\equiv c+1 \pmod 3.\end{aligned}$$

This completes the induction. It follows that $P_c(2021) \equiv 0 \pmod 3$ if and only if $c \equiv 2 \pmod 3$.

Solution 2: Extend $P_c(n)$ to the case $c = 0$ for convenience (in this case there are 0 cows always, so $P_c(n) = 1$ for all n). We first observe that for $c \geq 0$, $n \geq 1$, $P_c(n)$ is equivalently the number of sequences $c_1, c_2, \ldots, c_n$ where $c_1 = 1$ and $c_{i+1} \in [c_i, 2c_i]$ for all i. Here c_i is the number of cows on day i, and the number of ostriches on day i is derived as $2c_{i-1} - c_i$.

Then for $c \geq 1$ and $n \geq 1$, we have the recurrence

$$P_c(n+1) = P_{c-1}(n+1) - P_{c-1}(n) + P_{2c-1}(n) + P_{2c}(n),$$

by the following bijection: take a sequence starting with c and subtract 1 from the first number. Either it is a valid sequence starting with $c-1$, or the second number is $2c-1$, or the second number is $2c$. In the first case, this covers all valid sequence starting with $c-1$ other than those where the second number is $c-1$, so the number of such sequences is $P_{c-1}(n+1) - P_{c-1}(n)$.

From the above recurrence, we now claim that $P_c(n) \equiv c + 1 \bmod 3$ for all $n \geq 2$. (For $n = 1$, $P_c(1) = 1$ so it doesn't hold.) Induct on n. For the base case, $P_c(2)$ counting valid sequences of length 2, of which there are exactly $c+1$. For the inductive step, using the recurrence:

$$P_c(n+1) \equiv P_{c-1}(n+1) - (c) + (2c) + (2c+1) \equiv P_{c-1}(n+1) + 1,$$

and since $P_0(n+1) = 1$, the result follows.

5. We claim that each player has a strategy that prevents them from losing, hence if both players play optimally, then the game will go on infinitely. The strategy to prevent yourself from losing is simple: if you have 0 or 1 stones on your turn, then gain a stone (this is forced); otherwise if you have $n \geq 2$ stones, then always give away as many stones as possible ($\lfloor \frac{n}{2} \rfloor$) to the opponent.

   We argue that this strategy works for Gog; the argument that it works for Magog is identical. Suppose that it is our turn and we have not lost yet; then we have at most 19 stones, and after giving as many as possible away (or gaining one in the case of 0 or 1), we will have at most 10 stones. Then either the opponent loses immediately, or on their turn they can give us at most 9 stones. Since that leaves us with at most $10 + 9 = 19$ stones again on our turn, we do not lose and by induction, we can never lose.

6. Let the prime factorization of $n!$ be $p_1^{x_1} p_2^{x_2} p_3^{x_3} \cdots p_k^{x_k}$, where $p_1 = 2$, $p_2 = 3$, and so on are all the prime numbers between 1 and n, inclusive. The number of divisors of $n!$ is $(x_1+1)(x_2+1)(x_3+1)\cdots(x_k+1)$.

   We consider the following algorithm which assigns each value $x_i + 1$ with a distinct number between 1 and $2n$. Visit the prime numbers in order, starting with p_1 and ending with p_k. For each p_i, assign $(x_i + 1)$ to a multiple of $(x_i + 1)$ that has not yet been assigned. Assuming such a multiple always exists, in the end we have that each $(x_i + 1)$ divides the number it is assigned to, and the product of all assigned numbers divides $1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot (2n) = (2n)!$. Thus, it remains to show that at each step there is a multiple of $x_i + 1$ between 1 and $2n$ that is not assigned.

The number of factors of p_i going into $n!$ is given by the formula

$$x_i = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p_i^j} \right\rfloor,$$

so it can be bounded above:

$$< \sum_{j=1}^{\infty} \frac{n}{p_i^j} = n\frac{(1/p_i)}{1-(1/p_i)} = \frac{n}{p_i - 1}.$$

Therefore,

$$x_i + 1 < \frac{n + p_i - 1}{p_i - 1} \leq \frac{2n-1}{p_i - 1} < \frac{2n}{p_i - 1},$$

which implies there are at least $p_i - 1$ multiples of $(x_i + 1)$ between 1 and $2n$. On the other hand, the number of primes so far (p_1 through p_{i-1}) is at most $p_i - 2$, since 1 is not prime. So this completes the proof that there is always a multiple of $(x_i + 1)$ available.