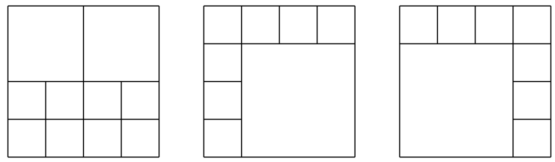
2019 Utah Math Olympiad March 23, 2019

1. How many ways can you divide a 4×4 square into a collection of one or more 1×1, 2×2, 3×3, and 4×4 squares? For example, three *different* ways are shown below.



2. Find all polynomials $p(x)$ with real coefficients such that $p(x)^2 = p(p(x))$.

3. Find all nonnegative integers a, b, c such that $2^a + 2^b = c!$.

4. A game is played between a "leader", a ball, and 10 "players" (not including the leader). One of the players is secretly the traitor. The leader begins with the ball, and passes it to one of the players. When a player receives the ball, they take a turn, which consists of passing the ball to another player (not themselves and not the leader). Players continue taking turns according to who has the ball.

   Before the game begins, the leader gives *instructions* to each player. (The leader is allowed to give different instructions to different players; see the example below.) The instructions consist of an infinite list of player names. When a player receives the ball for the 1st time, they pass the ball to the 1st player on their list; when they receive the ball for the 2nd time, they pass the ball to the 2nd player on their list, and so on. All players follow the instructions exactly, except the traitor, who may ignore the instructions. (Assume that a player's instructions do not include their own name.)

Player 1 Instructions: 1. Pass to Player 7 2. Pass to Player 4 3. Pass to Player 8 ⋮	Player 2 Instructions: 1. Pass to Player 3 2. Pass to Player 4 3. Pass to Player 10 ⋮	Player 3 Instructions: 1. Pass to Player 5 2. Pass to Player 1 3. Pass to Player 2 ⋮	Player 4 Instructions: 1. Pass to Player 10 2. Pass to Player 5 3. Pass to Player 7 ⋮	Player 5 Instructions: 1. Pass to Player 7 2. Pass to Player 3 3. Pass to Player 9 ⋮
Player 6 Instructions: 1. Pass to Player 2 2. Pass to Player 10 3. Pass to Player 3 ⋮	Player 7 Instructions: 1. Pass to Player 2 2. Pass to Player 8 3. Pass to Player 2 ⋮	Player 8 Instructions: 1. Pass to Player 3 2. Pass to Player 9 3. Pass to Player 6 ⋮	Player 9 Instructions: 1. Pass to Player 4 2. Pass to Player 5 3. Pass to Player 8 ⋮	Player 10 Instructions: 1. Pass to Player 8 2. Pass to Player 5 3. Pass to Player 3 ⋮

   If the ball is passed to every player at least once (including the traitor), the game ends and the leader wins. On the other hand, if the ball is passed twice to the traitor, the game ends and the traitor wins. If the game goes on forever, the traitor also wins.

   Prove that the leader may give instructions to the players so that the leader wins no matter which player is the traitor and how the traitor plays; or prove that this is impossible.

5. Let I be the incenter of triangle ABC and D the point on $\overline{AB}$ tangent to the incircle. The line ID meets the circumcircle of ABI at I and E. If $IE = AC + BC$, what is angle C?

6. Prove that for all $n \geq 200$, if $n \equiv 0$ or $n \equiv 1 \bmod 4$, then it is possible to put n balls of various colors in a bag such that when two balls are drawn out (without replacement), there is an equal chance of the two colors being the same and the two colors being different.

   For example, this is possible when $n = 13$: the bag can contain 9 red balls, 3 green balls, and 1 blue ball.