2025 Utah Math Olympiad

April 5, 2025

1. A regular octagon and all of its diagonals are drawn. Find, with proof, the number of squares that appear in the resulting diagram. (The sides of each square must lie along one of the edges or diagonals of the octagon.)

2. Given polynomials $f(x)$ and $g(x)$, where $g(x)$ is not the zero polynomial, we define $\left\lfloor \frac{f(x)}{g(x)} \right\rfloor$ to be the unique polynomial $q(x)$ such that we can write $f(x) = g(x) \cdot q(x) + r(x)$, where $r(x)$ is a polynomial such that either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$. Find all polynomials $p(x)$ with real coefficients such that
$$\left\lfloor \frac{p(x)}{x} \right\rfloor + \left\lfloor \frac{p(x)}{x+1} \right\rfloor = x^2.$$

3. Menelaus and Paris are having an apple-picking competition. The two players take turns, starting with Menelaus. On Menelaus's turn, he can either pick apples or spoil all the apples in Paris' basket (discarding them). On Paris's turn, he can either pick apples or steal the apples in Menelaus's basket (placing them in his basket). However, Paris can only steal once during the competition. The first time each player picks apples, he picks one apple. Each subsequent time he picks apples, he picks one more apple than the previous time. The player that first gets 1225 unspoiled apples in his basket wins the competition.

   Here are the first five turns in a sample game, where after each player's action, an ordered pair (m, p) is listed, indicating the numbers of apples in each person's basket:

   Menelaus picks one apple: $(1, 0)$. Paris picks one apple: $(1, 1)$.

   Menelaus picks two apples: $(3, 1)$. Paris picks two apples: $(3, 3)$.

   Menelaus spoils Paris's apples: $(3, 0)$. Paris picks three apples: $(3, 3)$.

   Menelaus picks three apples: $(6, 3)$. Paris steals Menelaus's apples: $(0, 9)$.

   Menelaus spoils Paris's apples: $(0, 0)$. Paris picks four apples: $(0, 4)$.

   Determine, with proof, which player has a winning strategy.

4. Define the *split-sum* of a positive integer N to be the sum of the two numbers obtained by splitting the base-10 representation of N into two parts, where the second part has as many digits as, or one more digit than, the first part. For example, the split-sum of 2025 is 45 because $20 + 25 = 45$, the split-sum of 123 is $1 + 23 = 24$, and the split-sum of 6 is 6. Find the number of positive integers m satisfying:

   - m has at most 2025 digits.
   - Each digit of m is equal.
   - The split-sum of m^2 is m.

5. Given parallelogram $ABCD$, we construct equilateral triangle ABP such that P is on the same side of $\overline{AB}$ as C and D. It is given that $\overleftrightarrow{CP}$ intersects $\overleftrightarrow{DA}$ at Q. Prove that there exists a point R on $\overleftrightarrow{AB}$ such that $\triangle CQR$ is equilateral.

6. Call a positive rational number r a *friendly number* if there exists a positive rational number $s \neq r$ such that $r^r = s^s$. Find, with proof, the second smallest friendly number.