1. **Solution 1:** We have
$$b^2 \geq c^2 = a^2 - ab + b^2 = a(a-b) + b^2 \geq b^2,$$
and so we must have equality throughout. Thus $b^2 = c^2$, and $a(a-b) = 0$, so $b = c$ and $a = b$, hence, the triangle is equilateral. As we also have $a + b + c = 1$ we conclude that $(a, b, c) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We check that this satisfies all three properties.

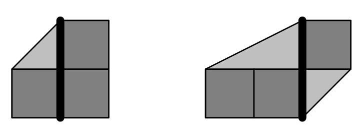
   Solution 2: Let α, β, γ be the opposite angles to a, b, and c. By the Law of Cosines,
$$a^2 - ab + b^2 = c^2 = a^2 + b^2 - 2ab\cos\gamma,$$
so $\cos\gamma = \frac{1}{2}$ and $\gamma = 60°$. Also, it is a well-known fact that in a triangle, the longest side is opposite the largest angle, and the shortest side is opposite the smallest angle. Since $a \geq b \geq c$, we deduce that $\alpha \geq \beta \geq \gamma$. Thus,
$$180° = \alpha + \beta + \gamma \geq 3\gamma = 180°,$$
and the equality implies $\alpha = \beta = \gamma = 60°$, so the triangle is equilateral. Hence $(a, b, c) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

   Solution 3: As in Solution 2, the Law of Cosines implies $\gamma = 60°$. Then by the Law of Sines, $a \geq b \geq c$ implies $\sin\alpha \geq \sin\beta \geq \sin\gamma$, hence $\sin\alpha \geq \frac{\sqrt{3}}{2}$ and $\sin\beta \geq \frac{\sqrt{3}}{2}$, hence $\alpha \geq 60°$ and $\beta \geq 60°$. Since the $\alpha + \beta + \gamma = 180°$, it follows that $\alpha = \beta = \gamma = 60°$ so $(a, b, c) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as before.

2. We claim that the area of P can be any number of the form $\frac{n}{2}$, where n is a positive integer and $n \neq 1, 3, 5$.

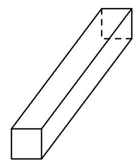
   First, note that if S consists of a $1 \times k$ block of unit squares, then it is already convex, and it has area k. Thus every number of the form $k = \frac{2k}{2}$ can be attained as the area of P. This implies that $n = 2k$ is achievable.

   If n is odd, we observe that the areas of the smallest convex polygons containing the arrangements of the following two groups of squares have areas $\frac{7}{2}$ and $\frac{9}{2}$, respectively.



   Note that for each arrangement, we can simply cut along the bolded line and add a $2 \times k$ block of unit squares in the middle, which will add $2k$ to the area. This shows that areas of the form $\frac{7}{2} + 2k = \frac{7+4k}{2}$ and $\frac{9}{2} + 2k = \frac{9+4k}{2}$ are attainable. It follows that n can be every odd integer greater than or equal to 7.

   We claim that $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ are not attainable as areas. Since we must use at least one unit square, then the area of P is at least 1, so $\frac{1}{2}$ is not attainable. Also, if we only use one unit square to construct P, then P will clearly only consist of that single unit square, which has area 1. Therefore to get an area exceeding 1, we must use at least two unit squares, so the area of P will be at least 2. Hence $\frac{3}{2}$ is not attainable. If we use three or more unit squares, then the area is at least 3, so if $\frac{5}{2}$ is attainable, then we must use exactly two unit squares to construct P. If the two squares are in the same row/column, then P will be a $1 \times k$ rectangle, and the area of P will be an integer. Thus we may assume that the two unit squares are not in the same row/column. In this case, we can dissect P into one unit square and two parallelograms as shown below.



If the second square is offset by (a, b) (where $a, b \neq 0$), then the parallelograms have area $|a| \geq 1$ and $|b| \geq 1$, and the area of the square is 1, so the area of P is at least $1 + 1 + 1 = 3$. It follows that P cannot have area $\frac{5}{2}$.

Thus the area of P can be anything of the form $\frac{n}{2}$, where n is a positive integer and $n \neq 1, 3, 5$.

3. **Solution 1:** Note that $2^{10} = 1024$, ending in 24, and $2^{20} = 1048576$. Also note that $24 \cdot 76 \equiv 24 \pmod{100}$.

For nonnegative integers k, let $P_k = 2^{10+20k}$. Then

$$P_k = 1024 \cdot 1048576^k \equiv 24 \cdot 76^k \equiv 24 \pmod{100}$$

by induction on k. Hence it suffices to show that P_k begins with 20 for infinitely many k.

Decompose P_k in modified scientific notation as

$$P_k = c_k \cdot 10^{e_k},$$

where e_k is an integer and $2.1 \leq c_k < 21$. It is a well known fact that such a decomposition exists and is unique. We wish to show that $c_k \geq 20$ infinitely often. We have

$$P_{k+1} = 1048576 \cdot P_k = (1.048576 c_k) \cdot 10^{e_k + 6}.$$

Now we split into two cases:

- If $c_k < \frac{21}{1.048576}$ then $1.048576 c_k < 21$, so $c_{k+1} = 1.048576 c_k$.
- If $c_k \geq \frac{21}{1.048576}$ then $21 \leq 1.048576 c_k < 210$ so $c_{k+1} = 0.1048576 c_k$.

Suppose for the sake of contradiction that $c_k \geq 20$ only finitely many times. Then there exists an integer K such that $c_k < 20$ for all $k \geq K$. Note that $\frac{21}{1.048576} > \frac{21}{1.05} = 20$; hence for all $k \geq K$, we have $c_k < \frac{21}{1.048576}$, so $c_{k+1} = 1.048576 c_k$. It follows that c_k grows exponentially, and in particular $c_k \geq 21$ for some k, contradicting our definition. Thus $c_k \geq 20$ infinitely often, and infinitely many corresponding P_k begin with 20 and end with 24.

It can be computed that the three smallest such numbers are 2^{310}, 2^{1270}, and 2^{2250}.

Solution 2: By Euler's totient theorem, $2^{20} \equiv 1 \bmod 25$. Also, $2^{10} = 1024 \equiv 24 \bmod 100$. Hence, we consider numbers of the form 2^{10+20k} for all $k \geq 0$. All such numbers are congruent to 24 both mod 25 and mod 4, and hence satisfy the condition on the last two digits.

Now, the first two digits of 2^{10+20k} are 20 if and only if it lies in the interval $[20 \cdot 10^n, 21 \cdot 10^n)$ for some n; taking the log base 10, this is true if and only if

$$(10 + 20k) \log 2 \in [n + \log 20, n + \log 21)$$

for some n, which rearranges to

$$(20 \log 2) \cdot k \in \left[(n-2) + \log \frac{2000}{1024}, (n-2) + \log \frac{2100}{1024}\right),$$

i.e., we need to show that the fractional part of ka, where $a = 20 \log 2$, is between $\log \frac{2000}{1024}$ and $\log \frac{2100}{1024}$ for infinitely many k.

To complete the proof, it is a known fact that for any irrational number a, the fractional parts of ka cover any subinterval of $[0, 1)$ infinitely many times — for completeness, we include this fact as a lemma below. Since $\log 2$ is irrational, a is also irrational, and we can apply the lemma. In particular, taking $r = \log \frac{2000}{1024}$ and $s = \log \frac{2100}{1024}$, the fractional parts of ka cover the desired subinterval for infinitely many choices of k.

Lemma: Let a be an irrational number and let $0 \leq r < s \leq 1$. Then the fractional part of ka lies in $[r, s)$ for infinitely many k.

Proof of lemma: Let $\{x\}$ denote the fractional part of x. First, we show that for any positive integer n, we can always pick values of k such that $\{ka\}$ lies in the interval $(0, \frac{1}{n})$. To do this, we split $(0, 1)$ into subintervals of the form $(0, \frac{1}{n})$, $(\frac{1}{n}, \frac{2}{n})$, $\ldots$, $(\frac{n-1}{n}, \frac{n}{n})$ (since a is irrational, $\{ka\}$ is never one of the endpoints of these intervals). By the Pigeonhole Principle, if we take $n + 1$ terms in the sequence $\{ka\}$, then there exist two terms $k_1 < k_2$ such that $\{k_1 a\}$ and $\{k_2 a\}$ lie in the same subinterval. Thus either $0 < \{(k_2 - k_1)a\} < \frac{1}{n}$ or $1 - \frac{1}{n} < \{(k_2 - k_1)a\} < 1$. We can pick n sufficiently large so that $\frac{1}{n} < s - r$.

- If $0 < \{(k_2 - k_1)a\} < \frac{1}{n}$, then so long as $t\{(k_2 - k_1)a\} < 1$, the multiples $\{t(k_2 - k_1)a\} = t\{(k_2 - k_1)a\}$ will form an arithmetic sequence with common difference less than $\frac{1}{n}$, and since $\frac{1}{n} < (s - r)$, clearly one of the terms $\{t(k_2 - k_1)a\}$ must lie in (r, s).
- If $1 - \frac{1}{n} < \{(k_2 - k_1)a\} < 1$, then so long as $(t - 1) < t\{(k_2 - k_1)a\} < t$, we can say that $\{t(k_2 - k_1)a\}$ will form a decreasing arithmetic sequence where the common difference has absolute value less than $\frac{1}{n}$. Since $\frac{1}{n} < s - r$, one of the terms $\{t(k_2 - k_1)a\}$ must lie in (r, s).

In either case, we can find a positive integer k such that $\{ka\} \in (r, s]$. From here, if we ignore the first k terms of the sequence, we can repeat the process, and we can do this forever, which gives us infinitely many possible values of k. $\square$

4. **Solution 1:** Let $x_1 = c$, and let $n = 1$ in the second equation. We see $x_m = y_m - c$. Substituting into our equations above, we see

$$y_{m+n} = y_m y_n + c,$$
$$y_{mn} = y_m + y_n - 2c.$$

In particular, letting $m = n = 1$ in the second equation, we have $y_1 = 2c$. If $c = 0$ then by induction $x_n = y_n = 0$ for all n. Otherwise, we compute

$$\begin{aligned} y_2 &= y_1^2 + c = c(4c+1), \\ y_3 &= y_1 y_2 + c = 2c^2(4c+1) + c = c(8c^2+2c+1), \text{ and} \\ y_4 - c &= y_1 y_3 = y_2^2, \text{ so} \\ 2(8c^2+2c+1) &= (4c+1)^2. \end{aligned}$$

Expanding this last equality yields $16c^2 + 4c + 2 = 16c^2 + 8c + 1$, and canceling like terms and solving for c, we obtain $c = \frac{1}{4}$. In this case,

$$y_{n+1} = \frac{1}{2} y_n + \frac{1}{4},$$

and by induction, $y_n = \frac{1}{2}$ and $x_n = \frac{1}{4}$ for all n. Thus the only solutions are $x_n = 0$ and $y_n = 0$ for all $n \geq 1$, and $x_n = \frac{1}{4}$ and $y_n = \frac{1}{2}$ for all $n \geq 1$.

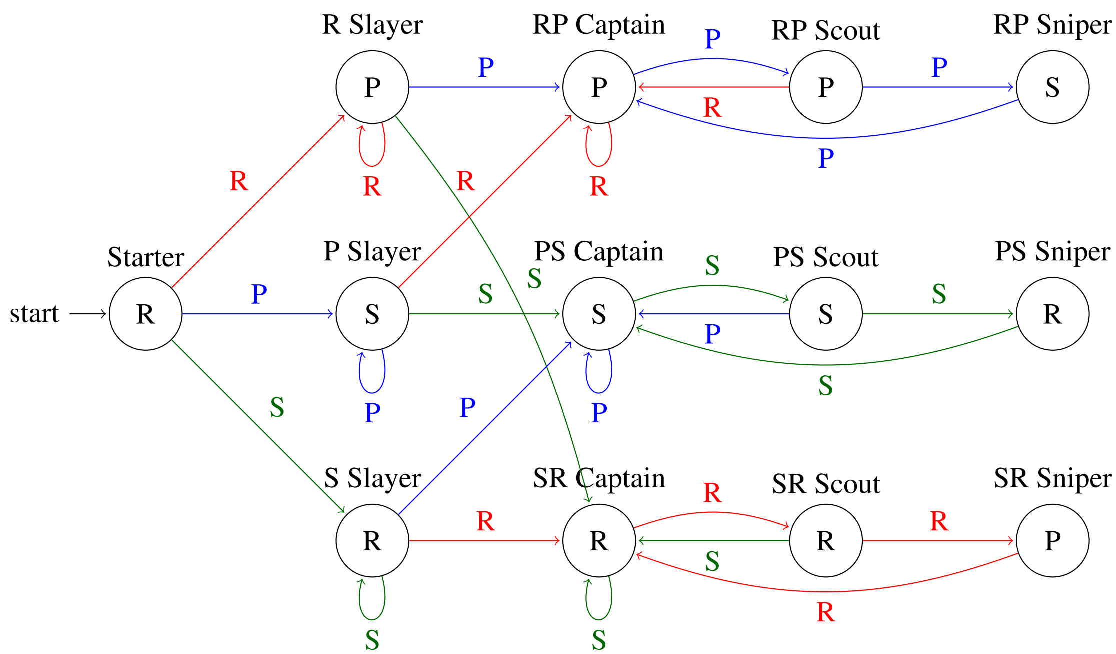
Solution 2: Note that $x_{n+1} = y_n y_1 = y_{n-1} y_2$ for $n \geq 2$. Dividing the two equal expressions for x_{n+1} (assuming they are nonzero), we find $1 = \frac{y_n y_1}{y_{n-1} y_2}$. Hence $\frac{y_n}{y_{n-1}} = \frac{y_2}{y_1}$ for all $n \geq 2$. It follows that y_n must be a geometric sequence, say $y_n = ar^{n-1}$ for all $n \geq 1$. Also, as $x_n = y_{n-1} y_1$ for all $n \geq 2$, we find $x_n = ar^{n-1} \cdot a = a^2 r^{n-1}$. Note that x_1 can be any value in order to satisfy the first equation, and it is easy to check that $y_n = ar^{n-1}$ (for $n \geq 1$) and $x_n = a^2 r^{n-1}$ (for $n \geq 2$) and $x_1 = k$ satisfies the first equation.

From the second equation with $n = 1$, we find that for $m \geq 2$, $y_m = x_m + x_1$, or $ar^{m-1} = a^2 r^{m-1} + k$. Thus $k = ar^{m-1}(1-a)$. If this is a constant for all $m \geq 2$, then either $a = 0$, $a = 1$, or else r^{m-1} does not change, in which case $r = 0$ or $r = 1$.

- If $a = 0$, then $x_n = y_n = 0$ for all positive integers n, and this satisfies the equation.
- If $a = 1$, then $k = 0$, so $x_1 = 0$. Hence $y_2 = 2x_1 = 0$, but $y_2 = ar = r$, so $r = 0$, which implies that $x_n = y_n = 0$ for all $n \geq 2$. But then $y_1 = x_1 + x_1 = 0$, contradicting $a = 1$.
- If $r = 0$, then as above, $x_n = y_n = 0$ for $n \geq 2$. Also, $0 = x_2 = y_1^2$ implies that $y_1 = 0$ and $y_1 = 2x_1$ implies that $x_1 = 0$, so both sequences are the zero sequence.
- If $r = 1$, then $x_n = a^2$ for $n \geq 2$ and $y_n = a$ for $n \geq 1$. Then $y_1 = 2x_1$ implies that $a = 2x_1$, so $x_1 = a/2$. Then $y_3 = x_1 + x_2$ implies that $a = (a/2) + a^2$, so $a^2 - a/2 = 0$. Hence $a(a - 1/2) = 0$, so $a = 0$ or $a = 1/2$. We previously dealt with $a = 0$. If $a = 1/2$, then $y_n = 1/2$ for $n \geq 1$ and $x_n = 1/4$ for $n \geq 2$, and $x_1 = a/2 = 1/4$. Thus $x_n = 1/4$ and $y_n = 1/2$ for all positive integers n. We can easily check that these sequences work.

It follows that there are only two pairs of sequences: $x_n = y_n = 0$ for all n, or $x_n = 1/4$ and $y_n = 1/2$ for all n.

5. We demonstrate a strategy for the red team where they will lose at most 2 matches, while winning at least 332 matches. The strategy uses 13 team members, and is illustrated below:



Without loss of generality (cycling the names of rock, paper, and scissors if necessary), we may assume that the first blue player plays rock. The first red player might lose this matchup (depending on their choice of rock, paper, or scissors), and they initially act somewhat naively: if the blue player plays rock, then they pick the ”Rock Slayer,” who plays paper and will always defeat a player who plays rock. The Rock Slayer picks themself if the blue team plays rock, so if the blue team does not play paper or scissors, then the red team will win all future matchups. However, if the blue team plays paper, then there is a tie, and it is known that the two players on the blue team play rock and paper, so the Rock Slayer picks the Rock-Paper Captain to iron out a better strategy. Similarly, if the blue team plays scissors, then the red team loses a match, and the Rock Slayer picks the Rock-Scissors Captain to iron out a better strategy.

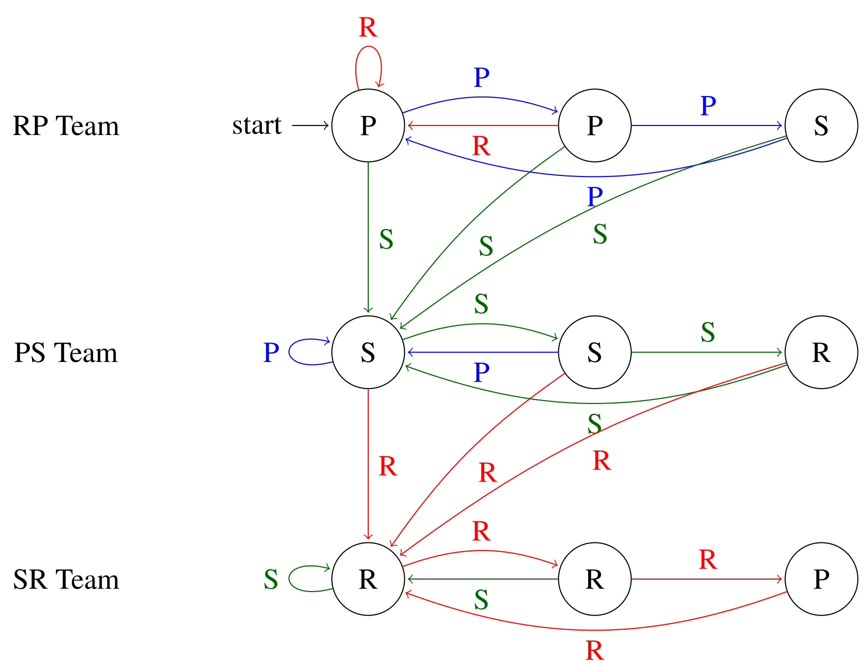
Now suppose that the red team knows that the blue team's two players pick different items; without loss of generality we may assume that they are rock and paper, so the Rock-Paper Captain will be chosen. The Rock-Paper Captain chooses paper, which can never lose to either blue team player. If the blue team plays rock, then the red team gains a point, and the Rock-Paper Captain picks themself for the next round. If the blue team plays paper, then neither team gets a point, and the Rock-Paper Captain responds by handing it off to the Rock-Paper Scout.

- The Rock-Paper Scout also plays paper (so it cannot lose), but it learns how the blue team's paper player responds to a paper-paper tie. If the blue team picks rock, then the red team gains a point, and the Rock-Paper Scout hands the next turn to the Rock-Paper Captain. If the blue team picks paper, then the red team knows that the blue team's paper player responds to a paper-paper tie by picking paper. Hence the red team knows that the blue team will pick paper on the next turn, so the Rock-Paper Scout hands the next turn to the Rock-Paper Sniper.
- The Rock-Paper Sniper plays scissors, which they know will beat the blue team's paper, gaining a point for the red team. Then the Rock-Paper Sniper hands the next turn to the Rock-Paper Captain.

By following this process, the red team will score a point at least once every three turns, and the blue team will never score a point beyond possibly 2 initial points.

In particular, there are 13 roles on the red team: Starter, Rock Slayer, Paper Slayer, Scissors Slayer, and then a three person Rock-Paper team, a three person Paper-Scissors team, and a three person Scissors-Rock team. Thus the red team can guarantee a victory against the blue team.

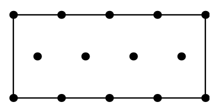
Note 1: It is also possible for the red team to win with just 9 players, by splitting them into three teams, where the first team never loses to a blue team that never plays scissors, the second team never loses to a blue team that never plays rock, and the third team never loses to a blue team that never plays paper. The following diagram illustrates such a strategy.



Note that each row functionally plays the same role as in our initial 13-person strategy. To see why this works, it can be shown that if the blue team only plays a single object (i.e., rock, paper, or scissors), then the red team will lose at most one match, while winning at least one out of every three of the remaining matches. We can also show that if during the course of the game, the blue team's players play two different objects, then they will eventually end up playing the appropriate team after winning at most two matches (i.e., a blue team with a rock player and a paper player will play the RP team, etc.).Then using the same reasoning as the 13-person strategy, we can show that the red team will never lose another match, while winning at least one out of every three matches.

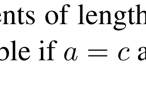
Note 2: The problem asks you to show that the red team can defeat the blue team by winning more points in 1000 rounds. In fact, a stronger statement turns out to be true: with 20 players, the red team can guarantee that, after a finite number of rounds, they win *every* round from that point on. However, the writers are not sure of the minimum number of players required for either version of the problem.

6. **Solution 1:** First, we claim that the vertices of the Type A and Type B triangles can only be placed on the triangular lattice below, where points on each row are 1 unit apart, and the rows are separated by distance $\sqrt{3}/2$.



Note that the left edge of the rectangle has length $\sqrt{3}$, and it must be covered by edges of the triangles, which have lengths 1 and $\sqrt{3}$. Clearly, the only way for it to be covered is if the $\sqrt{3}$ side of the Type B triangle coincides with the left edge. The same must be true for the right edge of the rectangle. Also, the bottom/top edges of the rectangle must be covered by a combination of sides of length 1 and $\sqrt{3}$, so $a + b\sqrt{3} = n$ for some integers a and b. But if $b \neq 0$, then $\sqrt{3} = \frac{n-a}{b}$, which is rational, a contradiction. Therefore, $b = 0$, which means that the bottom/top edges of the rectangle can only be covered by sides of length 1, and in each case, it is easy to see that these triangles' vertices will be on the above lattice.

Next, consider a line between pairs of tiles. We claim that for each 1-1-$\sqrt{3}$ triangle placed with an edge on the top or bottom of the rectangle, the $\sqrt{3}$ sides must match up exactly with $\sqrt{3}$ sides of other triangles. Otherwise, suppose that there is a line where two sides do not exactly match up. If this occurs, we can assume that one triangle "hangs" over the other triangle as shown below.

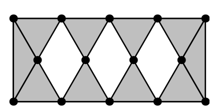


In order for the line to be fully covered on both sides, then any time one side "hangs" over the other, then the other side must have an additional triangle. Since all lines in the board have finite length, this cannot go on forever, so it follows that there is segment on the line where no hanging occurs—i.e., it starts and ends at the same time on both sides of the line. If there are a segments of length 1 and b segments of length $\sqrt{3}$ above, and c segments of length 1 and d segments of length $\sqrt{3}$ below, then $a + b\sqrt{3} = c + d\sqrt{3}$, and this is only possible if $a = c$ and $b = d$.

Now suppose that one of the triangles placed along the top/bottom is a 1-1-$\sqrt{3}$ triangle, and extend the line along its $\sqrt{3}$ side so it is bolded as shown below. If a $\sqrt{3}$ side is initially placed above this

$\sqrt{3}$ side (below left), then they can be thought of as covering two equilateral triangles, so it is as if there is an equilateral triangle the comes up from the bottom. Otherwise, the top side of the bolded line above the $\sqrt{3}$ side will initially be covered by a length 1 side. We observe that the intersection of the bolded line with the board has length $2\sqrt{3} \approx 3.46$. Since $\sqrt{3}+2 > 3.46$, it is impossible to place three triangles (of which one must use a $\sqrt{3}$ side) along this line, so we can only place a $\sqrt{3}$ and a 1 side along this line, with the $\sqrt{3}$ sides offset as shown below on the right.

However, now the top side of the upper 1-1-$\sqrt{3}$ triangle must align with an edge of one of the remaining triangles. This top edge is $(1+\sqrt{3})/2 \approx 1.366$ above the bottom of the rectangle, and the altitudes of the triangles have lengths $\sqrt{3}/2$ and $1/2$, so the top vertex of this additional triangle that aligns with the top side of the upper 1-1-$\sqrt{3}$ triangle must extend at least $(1+\sqrt{3})/2 + 1/2 \approx 1.866$ above the bottom edge of the rectangle. Since the rectangular board has height $\sqrt{3} \approx 1.732$, this will extend off the board, so it cannot be a valid tiling. Thus every 1-1-$\sqrt{3}$ triangle that is placed along the top/bottom triangle must be glued to a 1-1-$\sqrt{3}$ triangle along the $\sqrt{3}$ side, and it could be replaced by two equilateral triangles. Placing a equilateral triangles along the top/bottom edges, we obtain the following diagram, with holes that can be filled only by two equilateral triangles or two 1-1-$\sqrt{3}$ triangles (although if 1-1-$\sqrt{3}$ triangles are placed on the bottom/top line, the hole might be partially or totally filled with an equilateral triangle at the top or an equilateral triangle at the bottom).



In each case, however, we see that the only ways that the holes can be filled with tiles is if every tile must have all of its vertices placed on the triangular lattice.

Now let T_n be the number of tilings of the isosceles trapezoid of height $\sqrt{3}/2$ with bases of length n and $n-1$, and let P_n be the number of tilings of a parallelogram of length n and height $\sqrt{3}/2$.

Note that the top/bottom edges must be covered by edges of length 1, and by our previous argument, any 1-1-$\sqrt{3}$ triangle with a 1 edge along the top/bottom must be glued to a second 1-1-$\sqrt{3}$ triangle along the $\sqrt{3}$ side. Thus we can instead think of this as tiling with equilateral triangles of side length 1 and parallelograms that cover two such equilateral triangles. To compute T_n, we note that the tiling can end in an equilateral triangle, in which case the rest of the board can be tiled in P_{n-1} ways, or it can end in a parallelogram, in which case the rest of the board can be tiled in T_{n-1} ways. Thus

$$T_n = P_{n-1} + T_{n-1}. \tag{1}$$

To compute P_n, we note that the tiling can end in an equilateral triangle, in which case the board can be tiled in T_n ways, or it can end in a parallelogram, in which case the board can be tiled in P_{n-1} ways. Hence

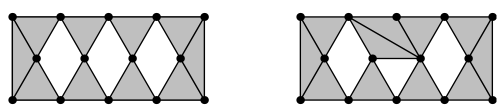
$$P_n = T_n + P_{n-1}. \tag{2}$$

From (1), we find $P_{n-1} = T_n - T_{n-1}$, which we substitute into (2) to find $T_{n+1} - T_n = T_n + (T_n - T_{n-1})$. Hence $T_{n+1} = 3T_n - T_{n-1}$, where $T_1 = 1$ and $T_2 = 3$. It's not hard to show that $T_n = F_{2n}$, the $2n$th Fibonacci number, because $T_1 = F_2$, $T_2 = F_4$, and

$$F_{2n} = F_{2n-1} + F_{2n-2} = (F_{2n-2} + F_{2n-3}) + F_{2n-2} = 2F_{2n-2} + F_{2n-3}.$$

Then $F_{2n-3} = F_{2n-2} - F_{2n-4}$, so $F_{2n} = 3F_{2n-2} - F_{2n-4}$, so F_{2n} and T_n satisfy the same initial conditions and recurrence relation, so they must be equal by induction.

Let x_n be the number of tilings of the n-board. Then we know that all triangles have vertices on our triangular grid. As mentioned above, each tile with an edge along the top/bottom edge must either come as an equilateral triangle, or as two 1-1-$\sqrt{3}$ triangles glued together to form a parallelogram, and this can create partial tilings shown below.



Consider the unshaded parallelograms in the diagram on the right. If left uncovered by the top/bottom edge triangles, it can either be covered by two 1-1-$\sqrt{3}$ triangles with the $\sqrt{3}$ sides coinciding with the vertical diagonal of the parallelogram, or it can be be covered by two equilateral triangles whose edges coincide with the horizontal diagonal of the parallelogram. On the other hand, the parallelogram could also have its top or bottom equilateral triangle be covered by 1-1-$\sqrt{3}$ triangles emanating from the top/bottom edges (as shown above on the right), in which case the horizontal diagonal of the parallelogram will coincide with one of the edges of the tile. However, in every case, each unshaded parallelogram must have exactly one of its diagonals coincide with an edge of a tile. If the vertical diagonal appears as an edge of a tile, then it cuts the grid into two smaller rectangular grids. In particular, if we do casework based on the first time a vertical diagonal appears, then prior to the first vertical diagonal, only horizontal diagonals can appear. If the rectangle prior to the first vertical edge has length k, then the top and bottom regions are trapezoids of length k, and each space can be tiled in T_k ways. It follows that for $n \geq 1$,

$$x_n = T_1^2 \cdot x_{n-1} + T_2^2 \cdot x_{n-2} + T_3^2 \cdot x_{n-3} + \cdots + T_n^2 \cdot x_0, \tag{3}$$

where by convention, $x_0 = 1$.

We claim that T_n^2 satisfies the recurrence

$$T_n^2 = 8T_{n-1}^2 - 8T_{n-2}^2 + T_{n-3}^2. \tag{4}$$

Recall that $T_n = 3T_{n-1} - T_{n-2}$, so $T_n^2 = 9T_{n-1}^2 - 6T_{n-1}T_{n-2} + T_{n-2}^2$. Applying the recurrence repeatedly, we find

$$\begin{aligned}
T_n^2 &= 9T_{n-1}^2 - 6T_{n-1}T_{n-2} + T_{n-2}^2 \\
&= 9T_{n-1}^2 - 6T_{n-1} \cdot \left(\frac{T_{n-1} + T_{n-3}}{3}\right) + T_{n-2}^2 \\
&= 7T_{n-1}^2 - 2T_{n-1}T_{n-3} + T_{n-2}^2 \\
&= 7T_{n-1}^2 - 2(3T_{n-2} - T_{n-3})T_{n-3} + T_{n-2}^2 \\
&= 7T_{n-1}^2 + T_{n-2}^2 + 2T_{n-3}^2 - 6T_{n-2}T_{n-3} \\
&= 7T_{n-1}^2 + T_{n-2}^2 + T_{n-3}^2 + (3T_{n-2} - T_{n-1})^2 - 6T_{n-2}T_{n-3} \\
&= 8T_{n-1}^2 + 10T_{n-2}^2 + T_{n-3}^2 - 6T_{n-2}(T_{n-1} + T_{n-3}) \\
&= 8T_{n-1}^2 + 10T_{n-2}^2 + T_{n-3}^2 - 18T_{n-2}^2 \\
&= 8T_{n-1}^2 - 8T_{n-2}^2 + T_{n-3}^2
\end{aligned}$$

This proves (4). We can use (4) this on iterations of (3):

$$\begin{aligned}
x_n &= T_1^2 \cdot x_{n-1} + T_2^2 \cdot x_{n-2} + T_3^2 \cdot x_{n-3} + T_4^2 \cdot x_{n-4} + \cdots + T_n^2 \cdot x_0 \\
-8(x_{n-1} &= \qquad\qquad T_1^2 \cdot x_{n-2} + T_2^2 \cdot x_{n-3} + T_3^2 \cdot x_{n-4} + \cdots + T_{n-1}^2 \cdot x_0) \\
+8(x_{n-2} &= \qquad\qquad\qquad\qquad T_1^2 \cdot x_{n-3} + T_2^2 \cdot x_{n-4} + \cdots + T_{n-2}^2 \cdot x_0) \\
-(x_{n-3} &= \qquad\qquad\qquad\qquad\qquad\qquad T_1^2 \cdot x_{n-4} + \cdots + T_{n-3}^2 \cdot x_0).
\end{aligned}$$

Note that this assumes that $n - 3 \geq 1$, i.e., $n \geq 4$. Adding these, we find

$$x_n - 8x_{n-1} + 8x_{n-2} - x_{n-3} = x_{n-1} + x_{n-2}.$$

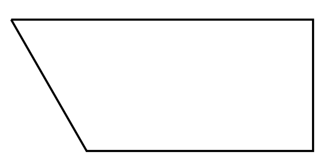
Therefore, for all $n \geq 4$,

$$x_n = 9x_{n-1} - 7x_{n-2} + x_{n-3}.$$

To compute the initial conditions, we can use the fact that $T_1 = 1$, $T_2 = 3$, and $T_3 = 8$, as well as $x_1 = 1$ and equation (3) to show $x_2 = 1^2 \cdot 1 + 3^2 \cdot 1 = 10$ and $x_3 = 1^2 \cdot 10 + 3^2 \cdot 1 + 8^2 \cdot 1 = 83$.

Solution 2: As in Solution 1, we assume that all tiles are placed such that they have vertices on the triangular grid.

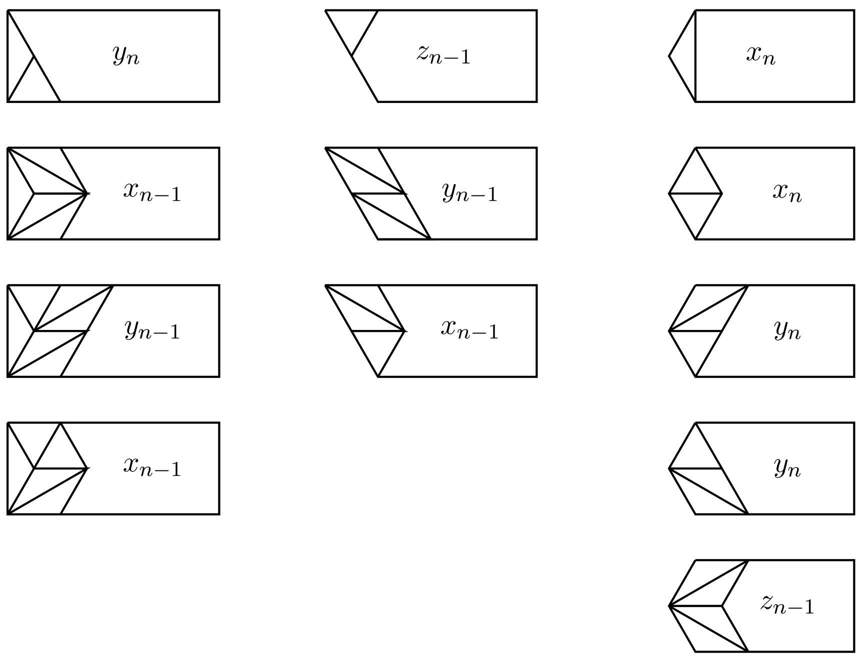
Let x_n be the number of tilings of the n-board. Let y_n be the number of tilings of boards of the type



where the top edge has length n. Let z_n be the number of tilings of boards of the type

where the top and bottom edges have length n. Note that x_n is also the number of ways to tile an $n \times \sqrt{3}$ rectangle if we remove a Type B triangle whose $\sqrt{3}$ side coincides with the left side of the board, because any tiling of an $n \times \sqrt{3}$ rectangle must start by placing a Type B triangle in this position.

In the below diagram, the first, second, and third columns represent the different ways that tilings can start for x_n, y_n, and z_n.



For x_n (the first column), we note that the left edge is always covered by a Type B tile. If the left-most edge on the bottom is covered by a Type A tile, we get the first case (y_n tilings). Otherwise, it must be covered by a Type B tile, which must be paired with another Type B tile to form a parallelogram. Next, the left edge on the top is either covered by a Type B triangle, paired with another Type B triangle to form a parallelogram (x_{n-1} tilings), or it is covered by a Type A triangle. After placing this, the left-most empty space can either be covered by two Type B triangles (y_{n-1} tilings) or a Type A triangle (x_{n-1} tilings). Hence

$$x_n = 2x_{n-1} + y_n + y_{n-1}. \tag{1}$$

For y_n (the second column), we see that the top left edge can either be covered by a Type A triangle (z_{n-1} tilings), or a Type B triangle, paired with another Type B triangle to form a parallelogram. In this latter case, the left-most empty space can either be covered by two Type B triangles (y_{n-1} tilings) or a Type A triangle (x_{n-1} tilings). Hence

$$y_n = x_{n-1} + y_{n-1} + z_{n-1}. \tag{2}$$

For z_n (the third column), we see that the left-most vertex can either be covered by the 120° angle in a Type B triangle, or else the horizontal edge emanating from this vertex will be uncovered. From there, we do casework based on the tiles above and below this horizontal edge: either A and A (x_n tilings), B and A (y_n tilings), A and B (y_n tilings), or B and B (z_{n-1} tilings). Hence

$$z_n = 2x_n + 2y_n + z_{n-1}. \tag{3}$$

Substituting (2) into (1), we find

$$x_n = 3x_{n-1} + 2y_{n-1} + z_{n-1}. \tag{4}$$

We can use (2) and (4) to iterate the right hand side of (3), finding

$$\begin{aligned} z_n &= 2(3x_{n-1} + 2y_{n-1} + z_{n-1}) + 2(x_{n-1} + y_{n-1} + z_{n-1}) + z_{n-1} \\ &= 8x_{n-1} + 6y_{n-1} + 5z_{n-1}. \end{aligned} \tag{5}$$

Thus equations (2), (4), and (5) all have similar forms.

Now if we substitute (5) and (2) into (4), we find

$$x_n = 3x_{n-1} + 10x_{n-2} + 8y_{n-2} + 7z_{n-2}. \tag{6}$$

By (4), we know that $x_{n-1} - 3x_{n-2} = 2y_{n-2} + z_{n-2}$. Multiplying this by 4, we find $8y_{n-2} + 4z_{n-2} = 4x_{n-1} - 12x_{n-2}$. Substituting this into (6), we find

$$\begin{aligned} x_n &= 3x_{n-1} + 10x_{n-2} + (4x_{n-1} - 12x_{n-2}) + 3z_{n-2} \\ &= 7x_{n-1} - 2x_{n-2} + 3z_{n-2}. \end{aligned} \tag{7}$$

Now suppose that we write (7) for n and $n-1$, and then we subtract twice the latter equation. We find

$$\begin{array}{rcl} x_n &=& 7x_{n-1} - 2x_{n-2} + 3z_{n-2} \\ -2(x_{n-1} &=& \qquad\qquad 7x_{n-2} - 2x_{n-3} + 3z_{n-3}) \\ \hline x_n - 2x_{n-1} &=& 7x_{n-1} - 16x_{n-2} + 4x_{n-3} + 3(z_{n-2} - 2z_{n-3}). \end{array} \tag{8}$$

Subtracting 3 times (4) from (5), we find $z_n - 3x_n = -x_{n-1} + 2z_{n-1}$. Therefore, $z_n - 2z_{n-1} = 3x_n - x_{n-1}$. In particular, it follows that $z_{n-2} - 2z_{n-3} = 3x_{n-2} - x_{n-3}$. Substituting this into (8), we find

$$x_n - 2x_{n-1} = 7x_{n-1} - 16x_{n-2} + 4x_{n-3} + 3(3x_{n-2} - x_{n-3}).$$

Collecting like terms, we find

$$x_n = 9x_{n-1} - 7x_{n-2} + x_{n-3}.$$

Remark: One can alternatively use methods from linear algebra to derive the recurrence if we write the equations (2), (4), and (5) in matrix form as

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 8 & 6 & 5 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix}.$$

Thus if $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 8 & 6 & 5 \end{pmatrix}$, then we can diagonalize A as $A = X\Lambda X^{-1}$, where Λ is a diagonal matrix whose entries are the eigenvalues of A. (An *eigenvalue* is a number λ such that the equation $Av = \lambda v$ has a solution for a non-zero vector v.) Using this representation, we get

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = A^{n-1} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = X\Lambda^{n-1}X^{-1} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

Since X, X^{-1}, and (x_1, y_1, z_1) are constant matrices, it follows that x_n, y_n, and z_n are linear combinations of λ_1^{n-1}, λ_2^{n-1}, and λ_3^{n-1}, where λ_1, λ_2, and λ_3 are the eigenvalues of A. Next, to calculate the eigenvalues of A, these are the roots of the characteristic polynomial of A, given by $\det(A - \lambda I) = 0$. We find

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ 1 & 1-\lambda & 1 \\ 8 & 6 & 5-\lambda \end{vmatrix} = (3-\lambda)((1-\lambda)(5-\lambda) - 6) - 2((5-\lambda) - 8) + (6 - 8(1-\lambda)) = 0.$$

This simplifies to $-\lambda^3 + 9\lambda^2 - 7\lambda + 1 = 0$. We further modify this to the form $\lambda^3 = 9\lambda^2 - 7\lambda + 1$. By the theory of linear homogeneous recurrence sequences, any sequence that is a linear combination of λ_1^{n-1}, λ_2^{n-1}, and λ_3^{n-1} satisfies the linear recurrence whose coefficients match the coefficients of the characteristic polynomial, so x_n satisfies the recurrence

$$x_n = 9x_{n-1} - 7x_{n-2} + x_{n-3}.$$

(The same recurrence is satisfied by y_n and z_n.)