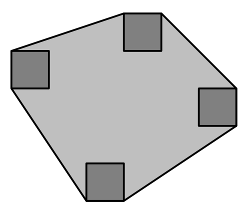
1. A triangle has side lengths a, b, and c satisfying the following three properties:

   - $a \geq b \geq c$;
   - $a^2 - ab + b^2 = c^2$;
   - $a + b + c = 1$.

   Find all possible values of (a, b, c).

2. One or more unit squares are placed in the plane, such that each square's vertices have integer coordinates. Let P be the smallest convex polygon that covers all the unit squares completely. For example, if there are four unit squares placed as shown below and darkly shaded, then P consists of the lightly shaded region plus the darkly shaded region:



   If the area of P is of the form $n/2$, where n is a positive integer, then what are all possible values of n?

   Note: It can be proven that the area of P is always of the form $n/2$, but you do not need to show this.

3. Show that infinitely many powers of 2 begin with 20 and end with 24.

4. Find all sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ of complex numbers such that for all positive integers m and n, we have

$$x_{m+n} = y_m y_n,$$
$$y_{mn} = x_m + x_n.$$

5. A red team with 27 players and a blue team with 2 players are competing in a rock-paper-scissors tournament. In each round of the tournament, one player from each team competes in a one-on-one game of rock-paper-scissors. The player who wins (if there is no tie) gets a point for their team. Then each of the two players either choose to keep playing, or select someone from their team to replace them. This is repeated for 1000 rounds.

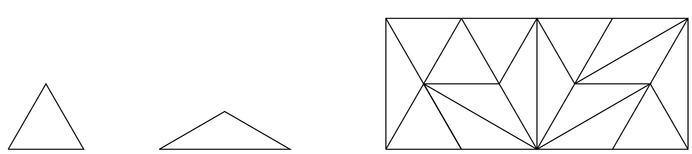
   Before the tournament begins, each team plans out their strategies in complete detail in advance, by doing the following: first, the team picks a player who will go first. To keep the strategies simple, each player will then commit to the following: which move they will play if they are selected to go up; and which player they will select to play next depending on whether the opponent's move is Rock, Paper, or Scissors. For example, if the Red team has 3 players, Alice, Bob, and Charlie, their strategy may look like this:

   - Alice goes first.
   - Alice: Play Rock. If the opponent's move is Rock, switch to Bob; otherwise switch to Charlie.
   - Bob: Play Scissors. Always switch to Alice.
   - Charlie: Play Rock. If the opponent's move is Rock, switch to Alice; If Paper, switch to Bob; If Scissors, stay with Charlie.

   Can the red team come up with a strategy to guarantee they win the tournament, no matter what strategy the blue team has planned?

   Note: In a one-on-one game of rock-paper-scissors, each player simultaneously picks one of Rock, Paper, or Scissors. If both players pick the same thing, the game ends in a tie. Otherwise, the winner is determined according to the following rules: Rock beats Scissors, Scissors beats Paper, and Paper beats Rock.

6. Type A triangles are equilateral with side length 1. Type B triangles are isosceles with side lengths 1, 1, and $\sqrt{3}$. Given an unlimited supply of Type A and Type B triangles, let x_n be the number of ways to tile a $n \times \sqrt{3}$ rectangle completely for a positive integer n. For example, one such tiling is shown below for $n = 4$.



   A *linear recurrence* for a sequence a_n is a formula for a_n, as a linear combination of finitely many previous terms in the sequence (a_{n-1}, a_{n-2}, a_{n-3}, and so on). For example, $a_n = 3a_{n-1} - 4a_{n-2}$ is a linear recurrence. Find a linear recurrence that is satisfied by the sequence x_n.