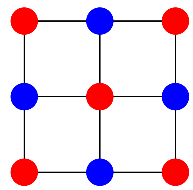
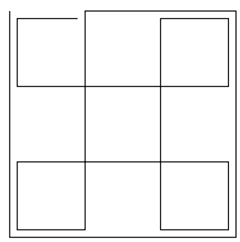
1. (a) A perfect path is impossible. Color the islands red and blue in a checkerboard pattern, as in the following picture:



Any path must alternate between red and blue islands. Therefore, the number of visits to red islands and the number of visits to blue islands must either be equal, or off by 1. However, since a perfect path must visit every island twice, it must have 10 visits to red islands and 8 visits to blue islands. So this is impossible.

(b) The following is one possible perfect path.



Alternatively, we can represent it using xy-coordinates:

$$(0,3) \to (0,0) \to (3,0) \to (3,3) \to (1,3) \to (1,0) \to (0,0) \to (0,1) \to (3,1) \to$$
$$(3,0) \to (2,0) \to (2,3) \to (3,3) \to (3,2) \to (0,2) \to (0,3) \to (1,3).$$

Note: In general, a perfect path is possible whenever n is even, but not when n is odd.

2. First, if $a^n = b^n$ for any odd n, then $a = b$, and we are done. So assume that $a^n \neq b^n$ for all odd n. Then for all odd n,

$$\frac{a^{2n} - b^{2n}}{a^n - b^n} = a^n + b^n$$

is the ratio of two rational numbers, and so is rational. This implies that a^n and b^n are rational, since

$$a^n = \frac{(a^n + b^n) + (a^n - b^n)}{2}$$

$$\text{and} \qquad b^n = \frac{(a^n + b^n) - (a^n - b^n)}{2}.$$

In summary, for all odd n we have that a^n and b^n are rational. Finally, since a^5 and a^3 are rational, $a = \frac{(a^3)^2}{a^5}$ must be rational (or $a = 0$, which is rational). Similarly for b. Thus, a and b are rational.

Note: It is not necessary to use all $n \geq 2$; for example, the above solution only really uses $n = 3, 5, 6$, and 10. There are other combinations of n that also work. In fact, $n = 3, 5$, and 6 is enough, but the proof is significantly more difficult. In general, it is possible to show that we need at least two odd values of n and at least one even value of n in order to obtain a solution.

3. **Solution 1:** The number of regions in the grid can be calculated in the following formula:

$$(\#\text{ regions}) = 4 - (\#\text{ adjacencies}) + (\#\text{ boxes}), \qquad (*)$$

where an *adjacency* is two filled squares that are horizontally or vertically adjacent, and a *box* is four filled squares in a 2×2 box. Why is this true? First, we overcount the number of regions by counting the number of filled squares, which is 4. Next, whenever two filled squares are adjacent, that combines two regions into one; so we subtract 1 for each adjacency. However, there is a case where this might now undercount: if when combining two regions into one, those were already the same region. This happens only if there is a 2×2 box, where we have four filled squares, then four adjacencies, so that the final adjacency is between two regions that were already combined. To correct for this, we add 1 if there is a 2×2 box in the figure.

Now from equation (*), we apply linearity of expectation to get:

$$\mathbb{E}[\#\text{ regions}] = 4 - \mathbb{E}[\#\text{ adjacencies}] + \mathbb{E}[\#\text{ boxes}].$$

(Here $\mathbb{E}[X]$ denotes the expected value of X; linearity of expectation states that for any X and Y, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.)

- To calculate $\mathbb{E}[\#\text{ adjacencies}]$, there are 12 possible adjacencies, and each adjacency happens with probability $\binom{7}{2}/\binom{9}{4}$, because there are $\binom{9}{4}$ ways to select four filled squares, and if two adjacent squares are filled in, the two remaining filled squares can be chosen in $\binom{7}{2}$ ways. So

$$\mathbb{E}[\#\text{ adjacencies}] = 12 \cdot \frac{\binom{7}{2}}{\binom{9}{4}} = 12 \cdot \frac{21}{126} = 2.$$

- To calculate $\mathbb{E}[\#\text{ boxes}]$, there are 4 ways to have a 2×2 box, so we have

$$\mathbb{E}[\#\text{ boxes}] = \frac{4}{\binom{9}{4}} = \frac{4}{126} = \frac{2}{63}.$$

Therefore, our final answer is

$$4 - 2 + \frac{2}{63} = \boxed{\frac{128}{63}}.$$

Solution 2: Alternatively, we can solve this using casework. In total, there are $\binom{9}{4} = 126$ ways to choose 4 squares out of nine. For the cases, we consider the number and sizes of the regions.

Case 1: One region, size 4.

There are various possible shapes here. For a 2×2 box, there are 4 ways. For an L-shape, 16 ways. For a T-shape, 8 ways. For a lightning bolt (zig-zag) shape, 8 ways. In total, $\boxed{36}$ ways.

Case 2: Two regions, sizes 3 and 1.

If the size 3 region is a line, there are 12 ways. If the size 3 region is an L-shape, there are 32 ways (4 corners to put the L in, and four rotations where the rotations have 1 way, 2 ways, 2 ways, and 3 ways). In total, there are $\boxed{44}$ ways in this case.

Case 3: Two regions, sizes 2 and 2.

Here the center square cannot be filled. There are two corners and two edges filled. There are $2 \cdot 4 = 8$ ways if the corners are opposite, and 4 ways if the corners are adjacent, for $\boxed{12}$ ways total.

Case 4: Three regions, sizes 2, 1, and 1.

If the center square is filled there are 4 ways. If not, there are 8 ways to place the region of size 2, then 3 ways for the remaining two regions, for 24 ways total. So $\boxed{28}$ ways in this case total.

Case 5: Four regions, sizes 1, 1, 1, 1.

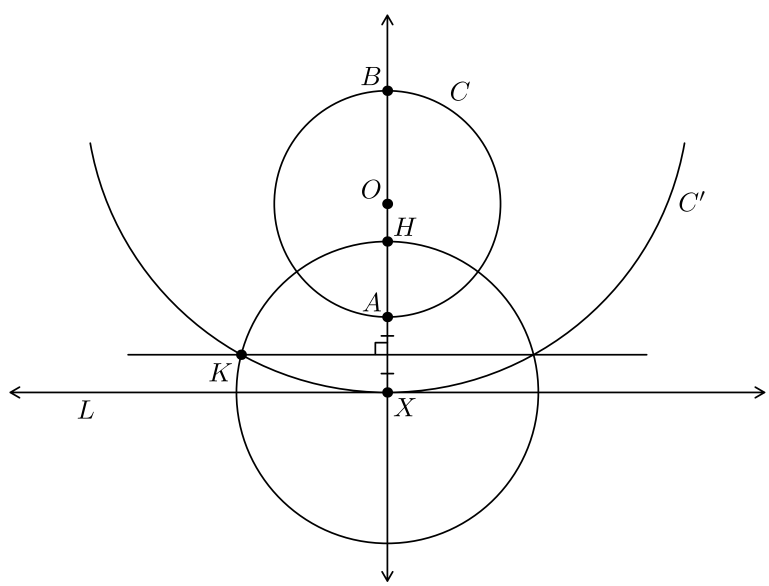
If the center is filled there are 4 ways; otherwise 2 ways, for $\boxed{6}$ total.

Double-check: We can verify that we did the casework correctly by adding up the total number of ways above. We get $36 + 44 + 12 + 28 + 6 = 126$, as expected.

Now to finish the problem, we compute the expected value:

$$\frac{(36) \cdot 1 + (44 + 12) \cdot 2 + (28) \cdot 3 + (6) \cdot 4}{126} = \frac{256}{126} = \boxed{\frac{128}{63}}.$$

4. **Solution 1:** Let O be the center of C. First, we draw a diagram:



Note that X, A, O, B are collinear (and lie in that order). Let $r = AO = OB$ be the radius of C, and let $XA = 2a$.

We first calculate the length of XK. Let K' be the midpoint of AX. By Pythagorean theorem,

$$XK^2 = K'K^2 + K'X^2 = K'K^2 + a^2.$$

Also by Pythagorean theorem,

$$K'K^2 = KB^2 - K'B^2 = XB^2 - K'B^2 = (2r+2a)^2 - (2r+a)^2 = 4ar + 3a^2.$$

Combining these,

$$XK^2 = (4ar + 3a^2) + a^2 = 4ar + 4a^2$$
$$\implies \quad XK = 2\sqrt{a^2 + ar}.$$

And $XH = XK$ by definition, so

$$XH = 2\sqrt{a^2 + ar}, \tag{1}$$

which is between $2a$ and $2a + r$. This confirms that H lies between A and O, as pictured in the diagram.

Let P be any point on line L. Let $x = XP$, and let C_P be the circle centered at P passing through H. The following is a standard fact:

Fact (perpendicular circles)**.** *Two circles are perpendicular if and only if $r_1^2 + r_2^2 = d^2$, where r_1, r_2 are the radii of the circles and d is the distance between the two centers.*

Using this fact, we will show that C_P is perpendicular to C. We need to calculate the radii of the two circles and the distance between the centers. First, the radius of C is

$$r_1 = r. \tag{2}$$

Second, the radius of C_P is PH, and by Pythagorean theorem this is

$$r_2^2 = XH^2 = XP^2 + XH^2 = x^2 + 4a^2 + 4ar, \tag{3}$$

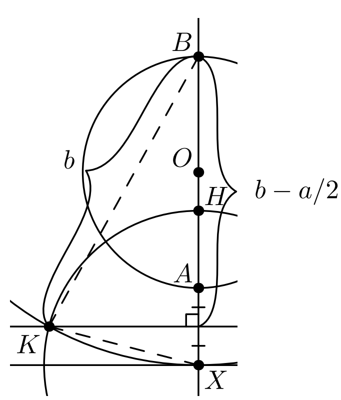
by (1). Third, the distance d is

$$d^2 = PO^2 = XP^2 + XO^2 = x^2 + (2a+r)^2 = x^2 + 4a^2 + 4ar + r^2. \tag{4}$$

Now we can see that adding (2) and (3), we get (4), so $r_1^2 + r_2^2 = d^2$, and we are done.

Solution 2: We solve this using coordinates. Let $X = (0,0)$, $A = (0,a)$, and $B = (0,b)$, so line L has equation $y = 0$.

If O is the center of circle C, then $O = (0, \frac{a+b}{2})$. We know K has y-coordinate $\frac{a}{2}$, so the difference between the y-coordinates of B and K is $b - a/2$.



Since K is on the circle of radius b centered at B, by the Pythagorean Theorem, it has x-coordinate $-\sqrt{b^2-(b-a/2)^2} = -\sqrt{ab-a^2/4}$. Therefore, as $K = (-\sqrt{ab-a^2/4}, a/2)$ and $X = (0,0)$,

$$KX = \sqrt{(ab-a^2/4)+a^2/4} = \sqrt{ab}.$$

It follows that $H = (0, \sqrt{ab})$.

Now let $P = (r, 0)$, and let $Q = (x, y)$ be a point where the circle through H centered at P intersects C. Then since Q lies on both circles, it must satisfy

$$x^2 + \left(y - \frac{a+b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2$$
$$(x-r)^2 + y^2 = PH^2 = ab + r^2.$$

Adding these equations, we find

$$2x^2 - 2rx + r^2 + 2y^2 - (a+b)y + (a+b)^2/4 = ab + (a-b)^2/4 + r^2.$$

This simplifies to

$$x^2 + y^2 - rx - (a+b)y/2 = 0. \tag{1}$$

On the other hand, note that the dot product of $\overrightarrow{PQ}$ and $\overrightarrow{OQ}$ is

$$\begin{aligned}\overrightarrow{PQ}\cdot\overrightarrow{OQ} &= (x-r, y)\cdot\left(x, y - \frac{a+b}{2}\right)\\ &= x^2 - rx + y^2 - (a+b)y/2\\ &= 0,\end{aligned}$$

where in the last step, we used (1). This implies that $\overrightarrow{PQ}$ and $\overrightarrow{OQ}$ are perpendicular, so the circles themselves must be perpendicular.

5. **Solution 1:** By the Law of Cosines and its converse, a triangle with side lengths a, b, c will have an angle of 120° between the sides with lengths a and b if and only if

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab\cos 120^\circ\\ &= a^2 + b^2 + ab.\end{aligned}$$

For any $m \geq 1$, set $a = 2m+1$, $b = 3m^2 + 2m$, and $c = 3m^2 + 3m + 1$. Note that $a < b < c$ and $a + b > c$, so this is a valid triangle. Also, we find that

$$\begin{aligned}a^2 + b^2 + ab &= (2m+1)^2 + (3m^2+2m)^2 + (2m+1)(3m^2+2m)\\ &= (4m^2+4m+1) + (9m^4+12m^3+4m^2) + (6m^3+7m^2+2m)\\ &= 9m^4 + 18m^3 + 15m^2 + 6m + 1\\ &= (3m^2+3m+1)^2\\ &= c^2,\end{aligned}$$

so this triangle is special.

Then, using the Euclidean algorithm we can compute

$$\gcd(b, c) = \gcd(3m^2 + 2m, m + 1) = \gcd(-m, m + 1) = 1.$$

Therefore, b and c share no common factor greater than 1, and so this triangle is primitive.

Note: This may seem like we pulled it out of a hat. However, there is some intuition behind it. It might make sense to see if we can generate a triple starting with an odd number $a = 2m + 1$ (much like there is an infinite family of Pythagorean triples $(2m+1, 2m^2+2m, 2m^2+2m+1)$ that generates $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, etc.). If there is a special triangle with $a = 2m + 1$, then

$$c^2 = (2m + 1)^2 + b^2 + (2m + 1)b.$$

Multiplying by 4 and completing the square, we find

$$(2c)^2 = 3(2m + 1)^2 + (2b + (2m + 1))^2.$$

Therefore, the difference between squares $(2c)^2$ and $(2b+(2m+1))^2$ is $3(2m+1)^2$, which is an odd number. In fact, since each odd number is the difference between consecutive squares, we find that

$$3(2m + 1)^2 = (6m^2 + 6m + 2)^2 - (6m^2 + 6m + 1)^2.$$

Setting $2c = 6m^2 + 6m + 2$ and $2b + (2m + 1) = 6m^2 + 6m + 1$, we obtain the triple that we initially stated.

Solution 2: Assume that $b = a + 2$ and that the angle between a and b is 120. We will show infinitely many solutions in this case. By the Law of Cosines, the triangle is special if and only if

$$(a + 2)^2 + a^2 + a(a + 2) = c^2$$

which rearranges to

$$c^2 - 3(a + 1)^2 = 1.$$

This is Pell's equation $x^2 - 3y^2 = 1$, so it has infinitely many solutions. For any such solution, set $c = x$ and $a = y - 1$ to get a special triangle (so $b = y + 1$).

For such a triangle, $\gcd(a, b, c)$ is at most 2, since $\gcd(a, b) = \gcd(y - 1, y + 1) = \gcd(y - 1, 2)$ divides 2. Since there are infinitely many solutions to Pell's equation, there are two cases: either there are infinitely many special a, b, c where $\gcd(a, b, c) = 1$, or there are infinitely many special a, b, c where $\gcd(a, b, c) = 2$. In the former case, we have infinitely many primitive special triangles and we are done. In the latter case, take the infinitely many special triangles and divide each side length by 2. This results in infinitely many distinct special triangles a', b', c' where $\gcd(a', b', c') = 1$.

6. The answer is that player 6 wins. This is trickier to prove than to state, and many intuitive justifications are not fully rigorous, so we have to be careful.

   Solution 1: First, we show that one of players $1, 2, 3, 4, 5$, or 6 must win. Second, we show that one of players $6, 7, 8, 9$, or 10 must win. Logically, it follows that 6 must win.

First part: We claim that, for $i = 1, 2, 3, 4, 5, 6$, if players 1 through $i - 1$ all vote for 6, then one of players i through 6 wins. The proof is by backwards induction. First, if players 1 through 5 all vote for 6, then player 6 can vote for themselves and win, and this is their top choice so they will do so. Next, if players 1 through 4 all vote for 6, then player 5 can either vote for 6 (making 6 win), or they can do better – and the only thing better for 5 is that 5 wins. Similarly, if players 1 through 3 vote for 6, then player 4 either makes 5 or 6 win by voting 6, or can do better, making 4 win. Continuing backwards until $i = 1$, we get that one of players $1, 2, 3, 4, 5$, or 6 must win.

Second part: We claim that, for $i = 6, 7, 8, 9, 10$, regardless of the first 5 votes, if players 6 through $i - 1$ vote for 10, then one of players i through 10 wins. The proof is by backwards induction, similar to before. For $i = 10$, we know that if players 6 through 9 vote for 10, then player 10 can vote for themselves and win. For $i = 9$, player 9 can either vote for 10, in which case player 10 wins, or do better, so either 9 or 10 must win. After continuing backwards until $i = 6$, we get that regardless of the votes of players 1 through 5, one of players 6 through 10 must win.

Solution 2: In this solution, we argue that the game is equivalent to a variant game where, on your turn instead of voting, you may "pass" your vote to the next player. Then we argue that it is rational for players 1 through 5 to pass their votes to player 6, and player 6 will then win.

First, we define the variant game. Each player starts with one vote. On your turn, you can either use all your votes, or you can "pass" your votes to the next player, who will then have your votes, plus one more. That player may then either use all their votes (in any distribution – for example you can vote for two different players), or pass all their votes to the next player, and so on.

We claim that the outcome of the variant game is exactly the same as the outcome in the original game. To see this, first imagine you are player 9 and you have some number of votes. Then instead of passing your votes on to the next player, since you know that player 10 is perfectly rational, you can just predict what player 10 would have voted, and vote that way yourself. So passing your votes to the next player never gives you any advantage, and we can just as well assume that player 9 does not pass on their votes. Similarly, going backwards, player 8 could just vote themselves what they know player 9 would do, instead of passing on their votes. So we can just as well assume that player 8 does not pass on their votes. Continuing in this way, we see that we can just as well assume that no player passes on their votes, and everyone is still behaving rationally in the variant game. That means that the variant game is equivalent to the original where passing was not allowed.

Now that we have defined the variant game, we argue that on any player's turn, *either that player can ensure that they win with a certain vote, or we can assume WLOG that they pass on their vote.* The idea is that, if you are player i you have various choices for what you can vote. Either one of these choices leads to player i winning, or else all of the choices lead to some player $j \neq i$ winning. In the latter case, *player $i + 1$'s preferences are exactly the same as player i's.* Therefore, player i is happy to pass on their vote to player $i + 1$, as they know that player $i + 1$ will make the same choice that they want.

At this point, we are assuming WLOG that each player's strategy is to pass on their vote, unless they have some way to win. But only one player can win. So in the actual outcome of the game, suppose player i wins. First we show $i \leq 5$ is impossible. In this case, players 1 through $i - 1$ pass on their vote (because they can't win), and player i gets i votes. After this, players $i+1$ through 10 pass on their vote (because they can't win either), so player 10 gets $10-i$ votes. But then player 10 can vote for themselves and win, which is a contradiction if $i \leq 5$.

Therefore, players 1 through 5 can't win, so all five of them pass on their votes. At this point, player 6 has 6 votes. So they can force themselves to win by voting for all 6 votes on themselves. This proves that player 6 wins in the variant game with our constraint on strategies; and by the logic above, that means that player 6 wins in the original game as well.