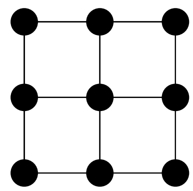
2020 Utah Math Olympiad March 21, 2020

1. An $n \times n$ grid of islands is connected by bridges, as in the following picture for $n = 3$:



In the above picture, there are 9 islands and 12 bridges. A *path* consists of starting at any island, travelling along the bridges, and ending at any island. For example, a path could visit the three islands along the top of the grid from left to right, crossing two bridges along the way. A *perfect path* is a path such that:

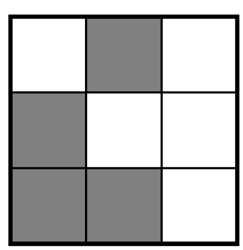
- Every island is visited exactly twice;
- Every bridge is crossed at least once; and
- The path never makes a U-turn, i.e. it never travels along a bridge and then immediately back again.

(a) Find a perfect path when $n = 3$, or prove it is impossible.

(b) Find a perfect path when $n = 4$, or prove it is impossible.

2. Let a and b be real numbers with the property that $a^n - b^n$ is rational for any positive integer $n \geq 2$. Show that either a and b are both rational, or that $a = b$.

3. In a 3×3 square grid, four of the nine squares are chosen at random and shaded. In the resulting figure, a *region* is a set of shaded squares that are vertically or horizontally (not diagonally) adjacent. For example, the following grid has two regions, one containing 3 squares and the other containing 1 square:

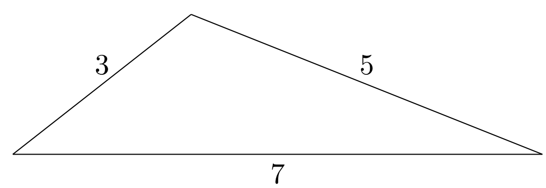


Find the expected value of the number of regions.

4. Consider a circle C and a line L which does not intersect C. Let A be the point on C nearest to L and let B be the point on C furthest from L; let line AB intersect L at point X. Let C' be the circle centered at B passing through X, and let K be an intersection of C' with the perpendicular bisector of line segment $\overline{AX}$. Finally, let the circle with center X passing through K intersect line segment $\overline{AB}$ at a point H. Prove that for every point P lying on L, the circle through H centered at P is perpendicular to C.

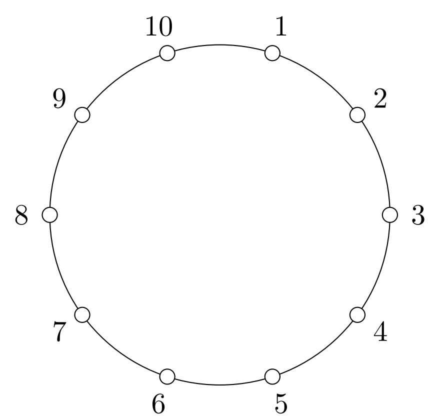
   Note: Two circles C_1 and C_2 are called *perpendicular* if they intersect at right angles. In other words, for each intersection point I of C_1 and C_2, if the tangent line is drawn to C_1 through I and to C_2 through I, the two tangents are perpendicular.

5. We say a triangle with integer side lengths a, b, and c is *primitive* if a, b, and c share no common factor greater than 1, and *special* if it has an angle with measure $120°$. For example, the following triangle with side lengths 3, 5, and 7 is both primitive and special:



   Prove that there are infinitely many primitive special triangles.

6. The positive integers between 1 and 10 are holding an election. They are sitting around a circular table – 1, then 2, then 3, and so on in clockwise order. Starting with 1 and going clockwise, each integer votes for a president (between 1 and 10). After all 10 integers have voted, the player with the most votes wins the election. The higher integer wins in case of tie.



   Every integer prefers itself to win; but if it can't win, it prefers the other integers in clockwise order from itself. For example, 8 prefers itself, then 9, then 10, then 1, then 2, and so on. Every integer is perfectly rational and knows that every other integer will behave perfectly rationally as well.

   Who wins the election? Prove your answer.