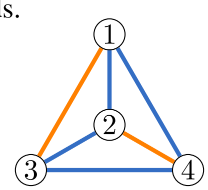
2017 Utah Math Olympiad Solutions March 18, 2017

1. There are 4 cities and exactly one road between every pair of cities. How many ways can you paint the roads orange and blue such that it is possible to travel between any two cities on only orange roads, and it is also possible to travel between any two cities on only blue roads? Prove your answer.

   For example, the following coloring of the roads would *not* be allowed, because there is no way to travel between 1 and 2 on orange roads.

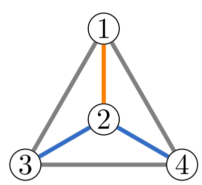
   

   Solution 1 Since there are only 6 roads, and the cities cannot be fully connected by 2 or fewer roads, we must paint 3 roads blue and 3 orange. The number of ways to do this is $\binom{6}{3} = 20$.

   Painting 3 roads of each color, it will always be possible to travel between any two cities on blue roads *unless* we paint the three blue roads in a triangle. Likewise for orange. It is impossible to have both a blue and an orange triangle. The number of possible triangles is $\binom{4}{3}$, and that triangle can be either orange or blue; this generates all incorrect colorings with 3 roads of each color.

   Thus, the number of ways to color the roads correctly is $\binom{6}{3} - 2\binom{4}{3} = 20 - 8 = \boxed{12}$.

   Solution 2 City 2's three roads cannot all be blue, or else it would not be connected by orange roads to the other cities. Similarly, its roads cannot all be orange. So it either has 2 blue and 1 orange, or 1 orange and 2 blue. There are 3 ways to have 2 blue and 1 orange, and 3 ways to have 2 orange and 1 blue, and by symmetry all these 6 cases are the same. So we can just count a particular case — assume $(1, 2)$ is orange and $(2, 3)$ and $(2, 4)$ are blue.

   If both $(1, 3)$ and $(1, 4)$ are orange, 1 isn't connected by blue, and if both are blue, 3 and 4 aren't connected to 1 and 2 by orange. Thus, one of them is orange and one of them is blue; either way, $(3, 4)$ is forced to be orange and the resulting picture is a valid coloring. So there are 2 valid colorings.

   Since there are 2 ways to complete the coloring in each of the six symmetric cases, the answer is $6 \times 2 = \boxed{12}$.

2. Quinn places a queen on an empty 8×8 chessboard. She keeps the board secret, but she tells Alex the row that the queen is in, and she tells Adrian the column. Then she asks Alex and Adrian, alternately, whether or not they know how many moves are available to the queen.

   Alex says, "I don't know."

   Adrian says, "I didn't know before, but now I know."

   Alex says, "Now I know, too."

How many moves must be available to the queen? (Note: A queen can move to any square in the same row, column, or diagonal on the chessboard, except for its current square.)

Solution First, we can label each square on the chessboard by how many moves the queen has if it is on that square, as shown below.

21	21	21	21	21	21	21	21
21	23	23	23	23	23	23	21
21	23	25	25	25	25	23	21
21	23	25	27	27	25	23	21
21	23	25	27	27	25	23	21
21	23	25	25	25	25	23	21
21	23	23	23	23	23	23	21
21	21	21	21	21	21	21	21

Suppose that Quinn told Alex that the queen is in row 1 or row 8. Then Alex would immediately know how many moves the queen has. Therefore, the only way to satisfy Alex's first statement is if the queen is in one of rows 2-7.

Using the first part of Adrian's statement, we find by the same logic that the queen must be in one of columns 2-7. So before Adrian heard Alex, Adrian knew which column the queen was in (one of columns 2-7), but after learning that the queen was in one of rows 2-7, Adrian knew how many moves the queen had. In more detail, we find the following:

- If Adrian knows that the queen is in column 1 or 8, then (as we just noted) Adrian would immediately know that the queen has 21 moves before Alex's statement, a contradiction.
- If Adrian knows that the queen is in column 2 or 7, then the queen could have either 21 or 23 moves available. However, when Adrian learns that the queen cannot be in rows 1 or 8, then Adrian finds that the queen must have 23 moves. This seems to fit with the given statements.
- If Adrian knows that the queen is in column 3 or 6, then the queen could have 21, 23, or 25 moves available. When Adrian learns that the queen cannot be in rows 1 or 8, Adrian is only able to conclude that the queen has either 23 or 25 moves, hence the second statement would not make sense in this case.
- If Adrian knows that the queen is in column 4 or 5, then the queen could have 21, 23, 25, or 27 moves available. When Adrian learns that the queen cannot be in rows 1 or 8, Adrian is only able to conclude that the queen has 23, 25, or 27 moves, hence the second statement would not make sense in this case.

Therefore, we see that the only way that the first two statements make sense are if the queen is in column 2 or column 7, in one of rows 2-7. Therefore, the queen has $\boxed{23}$ moves available.

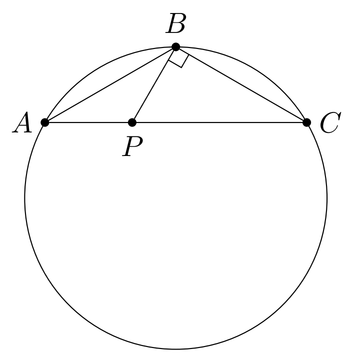
Note that the third statement is unnecessary, as Alex is simply confirming the what we know as observers.

3. Three distinct points A, B, and C lie on a circle with center O, such that $\angle AOB$ and $\angle BOC$ each measure 60 degrees. Point P lies on chord $\overline{AC}$ and the triangle BPC is drawn. If P is chosen so that it maximizes the length of the altitude from C to $\overline{BP}$, then determine the ratio $AP : PC$.

   Solution First we show that ABC is isosceles and $\angle ABC = 120°$. Since $\angle AOB$ and $\angle BOC$ are equal, chord lengths AB and BC are also equal. Also, $\triangle AOB$ and $\triangle BOC$ have equal sides $OA = OB = OC$ (the radius of the circle), and angles $\angle AOB = \angle BOC = 60°$, hence $\triangle AOB$ and $\triangle BOC$ are equilateral. Therefore, $\angle ABC = \angle ABO + \angle OBC = 60° + 60° = 120°$.

   Let F be the foot of the altitude from C to $\overline{BP}$. Since FC is the shortest distance from C to the line through B and P, and B is on that line, we can deduce that $FC \leq BC$, with equality if and only if $F = B$.

   Next we show that we can achieve $F = B$, so that $FC = BC$. Draw the line perpendicular to $\overline{BC}$ passing through B, and let P be the intersection of this line with $\overline{AC}$. Since $\angle ABC = 120°$, P lies between A and C. Choosing this P, we see that $\overline{BC}$ is perpendicular to $\overline{BP}$ by construction. Thus B is the foot of the altitude from C to $\overline{BP}$, so $F = B$. Thus we find that $FC = BC$ is achievable, so this P maximizes the length of the altitude from C to $\overline{BP}$. The location of P is unique because it is forced by $F = B$, and $F = B$ is the only way to maximize FC.



   Without loss of generality let $AB = BC = 1$, so that $AC = \sqrt{3}$ (by law of cosines, or splitting $\triangle ABC$ into two 30-60-90 triangles). Since $\triangle PBC$ is 30-60-90 with $BC = 1$, $PC = \frac{2}{3}\sqrt{3}$, which gives $AP = AC - PC = \frac{1}{3}\sqrt{3}$. Therefore, the desired ratio $AP : PC$ is $\boxed{1 : 2}$.

4. The polynomial $P(x)$ has integer coefficients, and $n\,P(n) \equiv 1 \pmod{16}$ for every odd positive integer n. That is, $nP(n)$ is 1 greater than a multiple of 16. Find, with proof, the minimum possible degree of $P(x)$.

   Solution 1 Define $Q(x) = xP(x) - 1$. If $P(x)$ has degree k, then $Q(x)$ has degree $k + 1$, with a constant term of -1. Therefore, we wish to find the minimal degree (greater than or equal to 1) of $Q(x)$ such that $Q(x)$ has a constant term of -1 and $Q(n) \equiv 0 \pmod{16}$ for all odd n.

   Taking

$$Q(x) = (x + 1)^2(x - 1), \tag{1}$$

then for all odd n, we see that $n+1$ and $n-1$ are even, and one of them is divisible by 4, hence $Q(n)$ is divisible by 16 for all odd n. Also, $Q(x)$ has a constant term of -1, so this $Q(x)$ works and has degree 3. Thus the minimum possible degree for Q is less than or equal to 3. It remains to rule out degree 1 and 2.

It is clear that $Q(x)$ cannot be linear, because if $Q(x) = ax - 1$, then $Q(1) = a - 1 \equiv 0 \pmod{16}$ implies $a \equiv 1 \pmod{16}$, and $Q(3) = 3a - 1 \equiv 0 \pmod{16}$ implies $a \equiv 11 \pmod{16}$, and these are contradictory.

If $Q(x)$ is quadratic, say $Q(x) = ax^2 + bx - 1$, then

$$Q(1) = a + b - 1 \equiv 0 \pmod{16} \quad \text{and} \quad Q(-1) = a - b - 1 \equiv 0 \pmod{16}.$$

Subtracting these, we find that $2b \equiv 0 \pmod{16}$. Adding the same equations, we find that $2a \equiv 2 \pmod{16}$. Also, we find that

$$Q(3) = 9a + 3b - 1 \equiv 0 \pmod{16}.$$

Subtracting the congruence for $Q(1)$, we find $8a + 2b \equiv 0 \pmod{16}$. But $2b \equiv 0 \pmod{16}$, hence $8a \equiv 0 \pmod{16}$. But as $2a \equiv 2 \pmod{16}$, we also know that $8a \equiv 8 \pmod{16}$. This is a contradiction, hence $Q(x)$ cannot be quadratic.

Therefore, the minimum possible degree for $Q(x)$ is 3, so the minimum possible degree for $P(x)$ is $\boxed{2}$. Using the relation $Q(x) = xP(x) - 1$ to solve for $P(x)$ from $Q(x)$ in (1), we find that this occurs when

$$P(x) = \frac{(x+1)^2(x-1)+1}{x} = x^2 + x - 1.$$

Solution 2 Note that if $nP(n) \equiv 1 \pmod{16}$, then $P(n) \equiv n^{-1} \pmod{16}$, the multiplicative inverse of n mod 16. Therefore, we can compute the following values for $P(n) \pmod{16}$.

n	1	3	5	7	9	11	13	15
$P(n)$	1	11	13	7	9	3	5	15

Using the method of finite differences, we know that if $P(x)$ has degree $k+1$, then $P(x+2) - P(x)$ has degree k. Therefore, we define the polynomial $P_0(x) = P(x)$, and in general, $P_{j+1}(x) = P_j(x+2) - P(x)$. If $P(x) = P_0(x)$ has degree $k+1$, then $P_{k+1}(x)$ must be constant, which implies that $P_{k+1}(x) \pmod{16}$ must also be constant. Using the values of $P(n)$ from the table above, we can compute values of $P_j(x) \pmod{16}$.

$P_0(n)$	1		11		13		7		9		3		5		15
$P_1(n)$		10		2		10		2		10		2		10	
$P_2(n)$			8		8		8		8		8		8		

As $P_1(x)$ is not constant mod 16, we see that it is impossible for P to have degree 1. But $P_2(x)$ is constant mod 16, so it looks like it might be possible for $P(x)$ to have degree 2. If $P(x)$ were to have degree 2, then $P_2(x)$ would be a constant integer, and from the above table it must be 8. Working backwards, we can fill in the above table with integers rather than residues, finding P_1 from P_2 and P_0 from P_1:

$P_0(n)$	1		11		29
$P_1(n)$		10		18	
$P_2(n)$			8		

Then we wish to find $P(x)$ such that $P(1) = 1$, $P(3) = 11$, and $P(5) = 29$. If $P(x) = ax^2 + bx + c$, then we find

$$\begin{aligned} a + b + c &= 1 \\ 9a + 3b + c &= 11 \\ 25a + 5b + c &= 29. \end{aligned}$$

Solving this system of equations, we find $a = 1$, $b = 1$, and $c = -1$. This suggests that $P(x) = x^2 + x - 1$ is such a polynomial. Note that

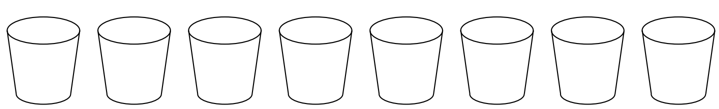
$$n(n^2 + n - 1) = (n + 1)^2(n - 1) + 1,$$

so if n is odd, then $n + 1$ and $n - 1$ are even, and at least one of them is a multiple of 4. Hence $(n + 1)^2(n - 1)$ is divisible by 16. Hence $nP(n) \equiv 1 \pmod{16}$ for all odd n, so we have a satisfactory polynomial of degree 2. Therefore, the minimum possible degree of P is $\boxed{2}$.

5. The Great Pumpkin challenges you to the following game. In front of you are 8 empty buckets. Each bucket is able to hold 2 liters of water. You and the Great Pumpkin take turns, with you going first.

   On your turn, you take 1 liter of water and distribute it among the buckets in any amounts you like. On the Great Pumpkin's turn, it empties two of the buckets of its choice. The Great Pumpkin is defeated if you cause any of the buckets to overflow.

   Given sufficiently many turns, can you defeat the Great Pumpkin no matter how it plays? Prove it, or prove that it is impossible.



Solution We can defeat the Great Pumpkin. Proceed in two phases.

Phase 1. Begin by pouring an equal amount, $\frac{1}{8}$, into every bucket. When the Great Pumpkin pours out two, refill those (taking $\frac{1}{4}$), and then distribute the remaining water ($\frac{3}{4}$) equally between all 8 buckets. On every turn, continue refilling the two buckets that the Great Pumpkin emptied, and distributing equally the remaining water.

Suppose that after our turn all buckets are at r and $r < \frac{1}{2}$. Then the Great Pumpkin empties out two buckets with r liters on its turn. On our next turn we take $2r$ to refill the two emptied buckets, and then distribute $1 - 2r$ equally, bringing all buckets to

$$r + \frac{1 - 2r}{8} = \frac{3}{4}(r) + \frac{1}{4}\left(\frac{1}{2}\right).$$

This is a weighted mean, and it expresses the number one-quarter of the way from r to $\frac{1}{2}$. Thus by repeating this strategy, the gap between r and $\frac{1}{2}$ will become smaller by a factor of $\frac{3}{4}$ each time. Since $\left(\frac{3}{4}\right)^n$ approaches 0, we can get as close as we want to $\frac{1}{2}$ in all buckets.

Stay in Phase 1 until the amount of the water in each bucket is greater than $\frac{7}{15}$ — this is still less than $\frac{1}{2}$, but it will be close enough to win. Then move on to Phase 2.

Phase 2. On your turn, six buckets have x liters of water in them, where $x > \frac{7}{15}$. Pick five of them, and fill in $\frac{1}{5}$ of a liter in each.

On your next turn, at least 3 buckets are left with $x + \frac{1}{5}$. Fill in $\frac{1}{3}$ of a liter in each.

Finally, on your following turn, at least 1 bucket is left with $x + \frac{1}{5} + \frac{1}{3}$. Fill the entire liter into this bucket.

The resulting amount of water in the bucket is

$$x + \frac{1}{5} + \frac{1}{3} + 1 = x + \frac{23}{15} > \frac{7}{15} + \frac{23}{15} = 2.$$

Since the bucket is only able to hold 2 liters, we have caused it to overflow and we win.

6. In a deck of n cards, there is one card of each number from 1 to n. Let a_n be the number of orderings of the deck such that the first card is less than the second card, the second card is greater than the third card, the third card is less than the fourth card, and so on. For example, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, and $a_4 = 5$. Determine, for all n, the remainder when a_n is divided by 4.

Solution 1 *Claim:* $a_1 \equiv 1$, $a_2 \equiv 1$, $a_3 \equiv 2$; for $n \geq 4$ even, $a_n \equiv 1 \pmod 4$; for $n \geq 5$ odd, $a_n \equiv 0 \pmod 4$.

Proof: Begin by noting that $a_1 = 1$, $a_2 = 1$, $a_3 = 2$. Now we have two cases, $n \geq 4$ even, and $n \geq 5$ odd.

Case 1: $n \geq 5$ **odd** The number of ways to order the deck, a_n, is equal to the number of ways to place numbers 1 through $n+1$ around a circular table with $n+1$ seats, such that every number is either less than its two neighbors or greater than its two neighbors, up to rotation. To see this, fix the place of $n+1$ around the table, and then put the deck in order clockwise, starting by placing the first card in the deck to the left of $n+1$, and ending by placing the last card in the deck to the right of $n+1$.

Now consider any valid placement of the numbers around the table, and instead of fixing the place of $n+1$, fix the place of 3, which is not equal to $1, 2$, n, or $n+1$. 1 and 2 cannot be adjacent around the table, or else 2 would not be higher than both neighbors; similarly n and $n+1$ cannot be adjacent around the table, or else n would not be lower than both neighbors. Since they are not adjacent, 1 and 2 can be switched and yield a valid ordering, and n and $n+1$ can be switched as well. Both possible switchings yeild 4 possibilities, meaning that every ordering around the table belongs to a set of 4 equivalent orderings when you do these two switches, and all 4 of these orderings are different since the positions of $1, 2, n$, and $n+1$ are different relative to 3. So the total number of orderings around the table is divisible by 4, i.e. $a_n \equiv 0 \pmod 4$.

Case 2: $n \geq 4$ **even** Pair up the cards into $n/2$ *pairs:* 1 and 2, 3 and 4, 5 and 6, and so on, up to $n-1$ and n. If the two cards in a pair, $2i-1$ and $2i$, are not adjacent in the deck, then they are interchangeable; we can switch them and we still get a valid ordering. Additionally, switching them does not change which pairs are adjacent. Thus considering all possible ways to switch the non-adjacent pairs, we see that a single ordering is part of a set of 2^k orderings, where k is the number of non-adjacent pairs.

This is divisible by 4 unless $k \leq 1$, so it remains to count orderings of the deck where at most one pair is non-adjacent. Say that only $2i-1$ and $2i$ are possibly not adjacent. In order for 1 and 2 to be adjacent, 2 must be higher than its neighbors, but it can only be higher than 1, so the ordering must end with $1, 2$. In order for 3 and 4 to be adjacent, in turn, 4 must be higher than its neighbors, so the ordering must end $3, 4, 1, 2$. This extends to all cards lower than $2i-1$ and $2i$, so we see that the ordering ends with $2i-3, 2i-2, \ldots, 3, 4, 1, 2$. On the other hand, in order for n and $n-1$ to be adjacent, $n-1$ must be lower than its neighbors, and a similar reasoning shows the ordering must begin with $n-1, n, n-3, n-2, \ldots, 2i+1, 2i+2$. The only way to then fill in the middle two cards is for $2i-1$ and $2i$ to be also adjacent, in the middle, in that order. So the entire deck ordering must be

$$n-1, n, n-3, n-2, n-5, n-4, \ldots, 5, 6, 3, 4, 1, 2.$$

Thus there is only 1 ordering left to count, namely, the one above. So $a_n \equiv 1 \pmod 4$.

Solution 2 Suppose that we define $a_0 = 1$. We claim that for $n \geq 0$,

$$a_n \equiv \begin{cases} 0 \pmod 4 & \text{if } n \geq 5 \text{ and } n \text{ odd}, \\ 1 \pmod 4 & \text{if } n = 1 \text{ or } n \text{ even}, \\ 2 \pmod 4 & \text{if } n = 3. \end{cases} \tag{1}$$

To prove this, we show that the following recursive relation must hold:

$$a_{n+1} = \binom{n}{1} a_1 a_{n-1} + \binom{n}{3} a_3 a_{n-3} + \binom{n}{5} a_5 a_{n-5} + \cdots. \tag{2}$$

First we prove (2). The condition we are given in the problem can be restated as "Every card in an odd position is smaller than its neighbors." In particular, this shows that the highest card (in this case, Card $n+1$) must appear in an even position. If Card $n+1$ appears in position $2k$, then there are $2k-1$ cards before it and $n-(2k-1)$ cards after it (possibly 0 cards). We can choose which cards appear before it in $\binom{n}{2k-1}$ ways, and we can arrange these $2k-1$ cards in a_{2k-1} ways. The remaining cards appear after Card $n+1$, and we see that they can be arranged in $a_{n-(2k-1)}$ ways. Therefore, the number of ways to arrange $n+1$ cards such that Card $n+1$ appears in position $2k$ is $\binom{n}{2k-1} a_{2k-1} a_{n-(2k-1)}$. Summing this over all k, we obtain the recursion in (2).

Now we prove (1) by strong induction. As a base case, observe that $a_1 = 1$, $a_2 = 1$, and $a_3 = 2$. Then suppose that (1) holds for all $n \leq 2k-1$, where $2k-1 \geq 3$. By (2),

$$a_{2k} = \binom{2k-1}{1} a_1 a_{2k-2} + \binom{2k-1}{3} a_3 a_{2k-4} + \cdots$$

As $a_{2j-1} \equiv 0 \pmod 4$ for $2j-1 \geq 5$, we know that the sum of all terms beyond these two terms will be $0 \pmod 4$. Therefore,

$$a_{2k} \equiv \binom{2k-1}{1} a_1 a_{2k-2} + \binom{2k-1}{3} a_3 a_{2k-4} \pmod 4$$

Also, $a_1 = 1$, $a_3 = 2$, and as $2k-2$ and $2k-4$ are even, we know $a_{2k-2} \equiv a_{2k-4} \equiv 1 \pmod 4$. Therefore,

$$\begin{aligned} a_{2k} &\equiv \binom{2k-1}{1} + \binom{2k-1}{3} 2 \pmod 4 \\ &= \frac{3(2k-1) + (2k-1)(2k-2)(2k-3)}{3} \\ &= \frac{(2k-1)(4k^2-10k+9)}{3} \\ &\equiv \frac{(2k-1)(2k+1)}{3} \pmod 4 \\ &= \frac{4k^2-1}{3} \\ &\equiv \frac{-1}{3} \pmod 4 \\ &\equiv 1 \pmod 4. \end{aligned}$$

Thus if (1) holds for all $n \leq 2k-1$, it must hold for $n = 2k$.

Similarly, suppose that (1) holds for all $n \leq 2k$, where $2k \geq 4$. Then by (2),

$$a_{2k+1} = \binom{2k}{1} a_1 a_{2k-1} + \binom{2k}{3} a_3 a_{2k-3} + \cdots. \tag{3}$$

If $2k+1 = 5$, then we can apply (3) to find that $a_5 = 16$. Similarly, if $2k+1 = 7$, we find $a_7 = 272$. In both cases, $a_{2k+1} \equiv 0 \pmod 4$. Otherwise, if $2k+1 \geq 9$, then in (3) we find that we will have at least four terms. In particular, in each product of the form $a_{2j-1}a_{2k-(2j-1)}$, at least one of $2j-1$ and $2k-(2j-1)$ will be greater than or equal to 5 (they are odd numbers summing to $2k = (2k+1)-1 \geq 9-1 = 8$). Therefore, at least one of a_{2j-1} and $a_{2k-(2j-1)}$ will be $0 \pmod 4$ by the inductive hypothesis in (1). Thus each of the terms $\binom{2k}{2j-1} a_{2j-1} a_{2k-(2j-1)}$ in (3) will be $0 \pmod 4$, hence

$$a_{2k+1} \equiv 0 \pmod 4.$$

Therefore, if (1) holds for all $n \leq 2k$ with $2k \geq 4$, it must hold for $n = 2k+1$.

Therefore, by induction, (1) must hold for all nonnegative integers n.