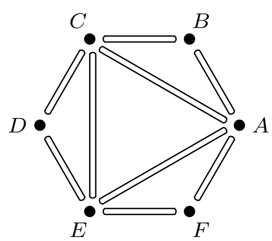
**2016 Utah Math Olympiad Solutions** March 19, 2016

1. Ada and Otto are engaged in a battle of wits. In front of them is a figure with six dots, and nine sticks are placed between pairs of dots as shown below. The dots are labeled $A$, $B$, $C$, $D$, $E$, $F$. Ada begins the game by placing a pebble on the dot of her choice. Then, starting with Ada and alternating turns, each player picks a stick adjacent to the pebble, moves the pebble to the dot at the other end of the stick, and then removes the stick from the figure. The game ends when there are no sticks adjacent to the pebble. The player who moves last wins. A sample game is described below. If both players play optimally, who will win?




Sample Game

1. Ada places the pebble at $B$.
2. Ada removes the stick $BC$, placing the pebble at $C$.
3. Otto removes the stick $CD$, placing the pebble at $D$.
4. Ada removes the stick $DE$, placing the pebble at $E$.
5. Otto removes the stick $EA$, placing the pebble at $A$.
6. Ada removes the stick $AB$ and wins.

**Solution** Ada will win. Here is one possible strategy.

Turn 1: Ada begins by putting the pebble at $A$.

Turn 2: Ada removes stick $AB$, and places the pebble at $B$.

Turn 3: The only remaining stick for Otto to choose is stick $BC$, so Otto removes this stick, placing the pebble at $C$.

Turn 4: Ada removes stick $CD$, placing the pebble at $D$.

Turn 5: Otto is again forced to choose the only remaining adjacent stick, $DE$. So Otto removes this stick, and places the pebble at $E$.

Turn 6: Ada removes stick $EF$, placing the pebble at $F$.

Turn 7: Otto is forced to remove stick $FA$, placing the pebble at $A$.

Turn 8: Ada removes stick $AC$, placing the pebble at $C$.

Turn 9: Otto is forced to remove stick $CE$, placing the pebble at $E$.

Turn 10: Ada removes stick $EA$, leaving no remaining sticks.

At this point, there are no valid moves, so Ada will win because every one of Otto's moves is forced.

2. Four fair six-sided dice are rolled. What is the probability that they can be divided into two pairs which sum to the same value? For example, a roll of $(1, 4, 6, 3)$ can be divided into $(1, 6)$ and $(4, 3)$, each of which sum to 7, but a roll of $(1, 1, 5, 2)$ cannot be divided into two pairs that sum to the same value.

**Solution 1** We split this into cases based on the the form of the unordered values shown on the dice. In the following, $1 \le a, b, c, d \le 6$ are distinct numbers. After determining the (unordered) numbers that can appear on the dice, we must determine how many ways we can order those dice rolls in sequence.

**Case 1:** The unordered numbers shown are $\{a, a, a, a\}$

In this case, it is clear that the pairing $(a, a)$ and $(a, a)$ will yield equal sums, so we have 6 possible choices for $a$, and this is the number of possible rolls that take this form.

**Case 2:** The unordered numbers shown are $\{a, a, a, b\}$

In this case, one of the pairs will contain $b$, hence the pairs will be $(a, b)$ and $(a, a)$. But these can never be equal as $a \neq b$.

**Case 3:** The unordered numbers shown are $\{a, a, b, b\}$

To obtain equal sums, we must split the numbers into the pairs $(a, b)$ and $(a, b)$. There are $\binom{6}{2} = 15$ ways to select $a$ and $b$, and there are $\binom{4}{2} = 6$ ways to order the rolls (select two of the rolls to be $a$'s). Hence there are $15 \cdot 6 = 90$ total rolls that take this form.

**Case 4:** The unordered numbers shown are $\{a, a, b, c\}$

In this case, either $b$ pairs with $a$ or $b$ pairs with $c$. If $b$ pairs with $a$, then $a + b = c + a$, which is impossible as $b \neq c$. Therefore, $b$ must pair with $c$, so $2a = b + c$. Therefore, $b + c$ is even. We may assume without loss of generality that $b \leq c$, hence the possibilities for $(b, c)$ are

$$(b, c) = (1, 3), (1, 5), (2, 4), (2, 6), (3, 5), (4, 6).$$

Therefore, there are six ways to choose $(b, c)$, and $a$ is automatically determined (it is the average of $b$ and $c$). Then there are $\frac{4!}{2!} = 12$ ways to order the rolls, hence there are $6 \cdot 12 = 72$ total rolls that take this form.

**Case 5:** The unordered numbers shown are $\{a, b, c, d\}$

In this case, the sum of the pairs must be representable in at two ways with distinct integers. We find

$$\begin{aligned} 5 &= 1 + 4 = 2 + 3 \\ 6 &= 1 + 5 = 2 + 4 \\ 7 &= 1 + 6 = 2 + 5 = 3 + 4 \\ 8 &= 2 + 6 = 3 + 5 \\ 9 &= 3 + 6 = 4 + 5. \end{aligned}$$

These are the only numbers that can be represented as the sum of two numbers in at least two ways using distinct numbers. In particular, for the numbers $5, 6, 8, 9$, we know immediately what the four numbers $a, b, c, d$ are. For 7, we must choose two of the three pairs, and we can do this in three ways. Therefore, the number of ways to choose $\{a, b, c, d\}$ is $1+1+1+1+3 = 7$. Then we can order the rolls in $4! = 24$ ways. Thus there are $7 \cdot 24 = 168$ total rolls that take this form.

Adding these together, we find $6 + 90 + 72 + 168 = 336$ possibilities, so the answer is $\frac{336}{6^4} = \frac{7}{27}$.

**Solution 2** Let the four numbers be $(a, b, c, d)$. Consider these three events:

I. $a+b=c+d$

II. $a+c=b+d$

III. $a+d=b+c$.

We want the probability of "I or II or III". By inclusion-exclusion, this is

$$\mathbb{P}[\text{I}]+\mathbb{P}[\text{II}]+\mathbb{P}[\text{III}]-\mathbb{P}[\text{I and II}]-\mathbb{P}[\text{I and III}]-\mathbb{P}[\text{II and III}]+\mathbb{P}[\text{I and II and III}].$$

(Here, $\mathbb{P}$ means the probability of an event.) Symmetry in the problem implies that I, II, and III have equal probability, as well as "I and II", "I and III", and "II and III". Therefore, the probability we are looking for is simply

$$3\mathbb{P}[\text{I}]-3\mathbb{P}[\text{I and II}]+\mathbb{P}[\text{I and II and III}]. \tag{1}$$

To compute $\mathbb{P}[1]$, split into cases based on the value of the sum $s=a+b$. We have

$$\begin{aligned}\mathbb{P}[1]=\mathbb{P}[a+b=c+d] &= \sum_{s=2}^{12}\mathbb{P}[a+b=c+d=s] \\ &= \sum_{s=2}^{12}\mathbb{P}[a+b=s]\cdot\mathbb{P}[c+d=s] \\ &= \frac{1^2+2^2+3^2+4^2+5^2+6^2+5^2+4^2+3^2+2^2+1^2}{6^4} \\ &= \frac{146}{6^4}.\end{aligned}$$

The other two are easier. If I and II both hold, then $b=c$ and $a=d$, and if $b=c$ and $a=d$ then I and II both hold. The probability of this is simply

$$\mathbb{P}[\text{I and II}]=\mathbb{P}[a=d \text{ and } b=c]=\mathbb{P}[a=d]\cdot\mathbb{P}[b=c]=\frac{1}{36}=\frac{36}{6^4}.$$

Finally, if I and II and III all hold, that means $a=b=c=d$, which can happen in exactly six ways, so

$$\mathbb{P}[\text{I and II and III}]=\mathbb{P}[a=b=c=d]=\frac{6}{6^4}.$$

From (1), the desired probability is thus

$$3\cdot\frac{146}{6^4}-3\cdot\frac{36}{6^4}+\frac{6}{6^4}=\frac{336}{6^4}=\frac{7}{27}.$$

3. Can each positive integer $1, 2, 3, \ldots$ be colored either red or blue, such that for all positive integers $a, b, c, d$ (not necessarily distinct), if $a+b+c=d$ then $a, b, c, d$ are not all the same color?

**Solution** We claim that no such coloring exists. For the sake of contradiction, assume that such a coloring exists. Without loss of generality, we may assume that 1 is colored red. In the string of steps below, if a number $n$ is known to be colored red, we denote it as $n^R$, and if it is known to be colored blue, we denote it as $n^B$. Each of the implications below follow from the fact that if all but one of the distinct numbers in $a+b+c=d$ are the same color, then the final number must be a different color.

$$1^R + 1^R + 1^R = 3 \Rightarrow 3 \text{ is blue.}$$
$$3^B + 3^B + 3^B = 9 \Rightarrow 9 \text{ is red.}$$
$$1^R + 4 + 4 = 9^R \Rightarrow 4 \text{ is blue.}$$

Now we have

$$1^R + 1^R + 9^R = 11 \Rightarrow 11 \text{ is blue} \qquad \text{AND} \qquad 3^B + 4^B + 4^B = 11 \Rightarrow 11 \text{ is red.}$$

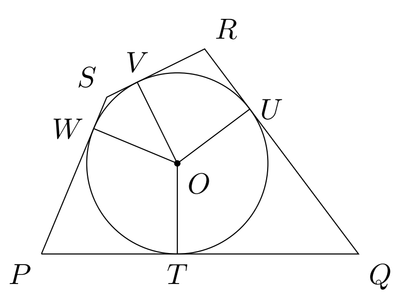
This is a contradiction because 11 can only have one color. Therefore, no such coloring exists.

4. Equiangular hexagon $ABCDEF$ has $AB = CD = EF$ and $AB > BC$. Segments $AD$ and $CF$ intersect at point $X$ and segments $BE$ and $CF$ intersect at point $Y$. If quadrilateral $ABYX$ can have a circle inscribed inside of it (meaning there exists a circle that is tangent to all four sides of the quadrilateral), then find $\frac{AB}{FA}$.

**Solution** We start by proving the following lemma about quadrilaterals that can have circles inscribed in them.

**Lemma 1.** *If quadrilateral $PQRS$ can have a circle inscribed inside of it, then $PQ + RS = QR + SP$. In other words, the sum of the opposite sides of such a quadrilateral are equal.*

*Proof.* Given a quadrilateral $PQRS$ that can have a circle inscribed in it, let $T, U, V, W$ be the points of tangency as shown below, and let $O$ be the center of the circle.

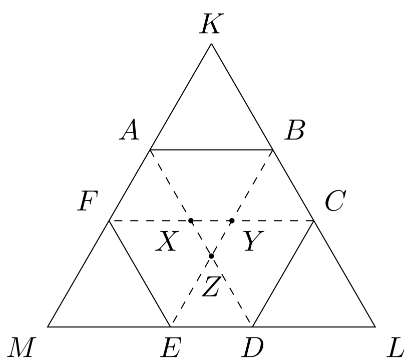



Then $\triangle OPT$ and $\triangle OPW$ are both right triangles with right angles at $T$ and $W$ respectively. They also have $OP = OP$ and $OT = OW$, so by the Pythagorean Theorem, $WP = PT$. Similarly, $TQ = QU$, $UR = RV$, and $VS = SW$. In fact,

$$PQ + RS = PT + TQ + RV + VS = WP + QU + UR + SW = PS + RQ.$$

Thus in any such quadrilateral, the sum of opposite sides is equal. □

We will use this lemma. But we first derive some other facts. To start, we extend line $AF$ through $A$ and $F$, we extend line $BC$ through $B$ and $C$, and we extend line $DE$ through $D$ and $E$. If the extensions of $AF$ And $BC$ meet at $K$, the extensions of $BC$ and $DE$ meet at $L$, and the extensions of $DE$ and $AF$ meet at $M$, then $\triangle KLM$ must be equilateral. This follows because hexagon $ABCDEF$ is equiangular, so $\angle FAB = \angle ABC = 120°$, and therefore, $\angle BAK = 180 - \angle FAB = 60°$, $\angle KBA = 180 - \angle ABC = 60°$, so $\angle K = 180 - \angle BAK - \angle KBA = 60°$. Similar arguments show $\angle L = \angle M = 60°$. We draw the diagram below, where $Z$ is the intersection of $AD$ and $BE$.




Note that $KB = AB$ and $CL = CD = AB$, hence $KL = 2AB+BC$. Similarly, $LM = 2AB+DE$ and $MK = 2AB + FA$. But triangle $KLM$ is equilateral, so $KL = LM = MK$, and thus $BC = DE = FA$. Let $x = AB = CD = EF$ and let $y = BC = DE = FA$. Then as $AM = MD = x+y$, we know that $\triangle AMD \sim \triangle KML$, hence $\triangle AMD$ is equilateral. In particular, $AD \parallel KL$. By similar logic, $FE \parallel KL$. We can apply the same argument to find

$$AB \parallel FC \parallel ML \text{ and } DC \parallel EB \parallel MK.$$

This means that all of the angles in the above diagram are 60° or 120°. It also implies that $ABYX$ is a trapezoid. It further implies $\triangle BAZ$ is equilateral, so $AZ = AB = x$. Also, $\triangle AFX$ is equilateral, so $AX = FA = y$.

As $x > y$, we find that $x = AZ \geq AX = y$, so $X$ lies between $A$ and $Z$. Therefore, $ZX = AZ-AX = x-y$. By symmetrical arguments, $XY = YZ = ZX = x-y$. As $\triangle BYC$ is equilateral, we also find $BY = BC = y$. Therefore, quadrilateral $ABYX$ is an isosceles trapezoid with $AB = x$, $BY = AX = y$, and $XY = x - y$. Applying Lemma 1, we find $AB + XY = BY + AX$, or $x + (x - y) = y + y$. Thus $2x = 3y$, and $\frac{AB}{FA} = \frac{x}{y} = \frac{3}{2}$.

5. Let $a_0, a_1, a_2, \dots$ be a sequence of integers (positive, negative, or zero) such that for all nonnegative integers $n$ and $k$,
$$a_{n+k}^2 - (2k+1)a_n a_{n+k} + (k^2+k)a_n^2 = k^2 - k.$$
Find all possible sequence $a_n$.

**Solution 1** We can factor the equation as

$$[a_{n+k} - ka_n][a_{n+k} - (k+1)a_n] = k(k-1). \tag{1}$$

Substituting in $k = 1$, we find that either $a_{n+1} = 2a_n$ or $a_{n+1} = a_n$ for all integers $n$.

If $p$ is a prime such that $p \mid a_r$, then by repeatedly applying $a_{n+1} = 2a_n$ or $a_{n+1} = a_n$, we find that $p$ divides $a_n$ for all integers $n \geq r$. Substituting $k = 2$ and $n = r$ into (1) we notice that the LHS is divisible by $p^2$, and the RHS is equal to 2; hence $p^2 \mid 2$, which is a contradiction. Thus $a_n$ is never divisible by any prime for any $n$, so $a_n = \pm 1$.

In particular, $a_{n+1} = a_n$ for all integers $n$, and $a_n$ is constant–either $1$ or $-1$. Substituting either of these sequences into (1) shows that both are valid solutions.

**Solution 2** After multiplying by 4 and completing the square twice, we find

$$(2a_{n+k} - (2k+1)a_n)^2 - (2k-1)^2 = a_n^2 - 1.$$

As the closest square to $(2k-1)^2$ is $(2k-2)^2$ (for $k$ positive), we see that if $x \neq 2k-1$ is an integer, then $|x^2 - (2k-1)^2| \geq 4k - 3$. Thus if $|a_n^2 - 1| \neq 0$, then $|a_n^2 - 1| \geq 4k - 3$ for all positive integers $k$. But this contradicts the fact that $|a_n^2 - 1|$ is finite. Therefore, $a_n^2 = 1$, and $|2a_{n+k} - (2k+1)a_n| = 2k - 1$. Now if $a_{n+k} = -a_n$, then $|2a_{n+k} - (2k+1)a_n| = 2k + 3$, which is not equal to $2k - 1$, hence $a_{n+k} \neq -a_n$. But as $a_n^2 = 1$ for all $n$, we see that either $a_n = 1$ for all $n$, or $a_n = -1$ for all $n$. It is easy to check that both of these sequences work.

**Solution 3** Factoring the equation as in (1) with $k = 1$, we find that either $a_{n+1} = a_n$ or $a_{n+1} = 2a_n$ for all nonnegative integers $n$. It follows that, for all nonnegative integers $n$, either (i) $a_{n+2} = a_n$, or (ii) $a_{n+2} = 2a_n$, or (iii) $a_{n+2} = 4a_n$. Now factor the equation with $k = 2$, and we get

$$(a_{n+2} - 2a_n)(a_{n+2} - 3a_n) = 2.$$

Fix $n \geq 0$. Consider the cases (i), (ii), and (iii). In case (ii), the above gives $0 = 2$, a contradiction. In case (i), the above gives $(-a_n)(-2a_n) = 2$, so $a_n^2 = 1$. In case (iii), $(2a_n)(a_n) = 2$, so $a_n^2 = 1$. In any case, $a_n^2 = 1$, so $a_n \in \{-1, 1\}$ for all $n$.

Because $a_{n+1}$ has to be either $a_n$ or $2a_n$, and $a_n \in \{-1, 1\}$ for all $n$, $a_n$ must be constant. Thus the only sequences that satisfy the desired property are $a_n = 1$ for all $n \geq 0$, and $a_n = -1$ for all $n \geq 0$.

6. Find all positive integer pairs $(u, m)$ such that $u + m^2$ is divisible by $um - 1$.

**Solution 1** If $u + m^2$ is divisible by $um - 1$, then

$$u(u + m^2) - m(um - 1) = u^2 + m$$

must also be divisible by $um - 1$. Therefore, if $(u, m)$ works, then $(m, u)$ works, so we may assume that $m \leq u$.

As $(um-1) \mid (u+m^2)$, we see that $um-1 \leq u+m^2$. Rearranging this, we find $u(m-1) \leq m^2+1$. Therefore, either $m = 1$ or $u \leq (m^2+1)/(m-1) = m+1+2/(m-1)$, which means $u$ is quite close to $m$. In particular, if $m \geq 4$, then $u \leq m+\frac{5}{3}$. But we assumed that $m \leq u$, hence $u \leq m+\frac{5}{3} \leq u+\frac{5}{3}$. The only integers $m$ that satisfy this are $m = u$ and $m = u-1$. If $m = u$, then $u^2-1$ divides $u+u^2$, and these numbers have a common factor of $u+1$, hence $u-1$ divides $u$. But $u \geq m \geq 4$, so this is impossible. If $m = u-1$, then $u(u-1)-1$ divides $u+(u-1)^2$, or $u^2-u-1$ divides $u^2-u+1$.

But when $u \geq 4$, $u^2 - u - 1 \geq 11$ (it's an increasing quadratic function). A number that is greater than 11 clearly cannot be a factor of a number that is two greater than that number. Therefore, this is impossible. Hence $m \leq 3$. We enumerate each of these possibilities separately.

- If $m = 1$, then $u - 1$ divides $u + 1$. This is only possible if $u = 2, 3$.
- If $m = 2$, then $2u - 1$ divides $u + 4$. Note that $2u - 1 > u + 4$ if $u > 5$. Hence $u \leq 5$. Testing the five possible values of $u$, we find that $u = 1, 2, 5$ work.
- If $m = 3$, then $3u - 1$ divides $u + 9$. Note that $3u - 1 > u + 9$ if $u > 5$. Hence $u \leq 5$. Testing the five possible values of $u$, we find that $u = 1, 5$ work.

We assumed that $m \leq u$, so we also must include the reversed pairs. Hence the only $(u, m)$ are

$$(1, 2), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 5), (5, 2), (5, 3).$$

**Solution 2** Suppose that $u + m^2 = (um - 1)q$ for some positive integer $q$. Then $m^2 - qum + u + q = 0$. By the quadratic formula,

$$m = \frac{qu \pm \sqrt{q^2u^2 - 4(u + q)}}{2}. \tag{1}$$

In order for this to be an integer, the discriminant, $q^2u^2 - 4(u + q)$, must be a perfect square. Note that $q^2u^2$ is a perfect square, and the next smallest perfect square is $(qu - 1)^2 = q^2u^2 - 2qu + 1$. For $u \geq 5$ and $q \geq 4$, we can squeeze $q^2u^2 - 4(u + q)$ in between $(qu - 1)^2$ and $(qu)^2$.

To show this, note that if $u \geq 5$ and $q \geq 4$, then certainly $(q - 2)(u - 2) \geq 5$. Hence

$$\begin{aligned}
(q - 2)(u - 2) &\geq 5 \\
qu - 2u - 2q + 4 &\geq 5 \\
2qu - 4u - 4q &\geq 2 \\
q^2u^2 - 4u - 4q &\geq q^2u^2 - 2qu + 2 = (qu - 1)^2 + 1.
\end{aligned}$$

Therefore, $(qu - 1)^2 < q^2u^2 - 4u - 4q < (qu)^2$. Hence if $u \geq 5$ and $q \geq 4$, then $q^2u^2 - 4u - 4q$ cannot be a perfect square, hence $um - 1$ cannot divide $u + m^2$. Therefore, either $u \leq 4$ or $q \leq 3$. We enumerate each of these cases below.

- If $u = 1$, then we want $(m - 1) \mid (1 + m^2)$. Clearly $(1, 1)$ does not work, while $(1, 2)$ and $(1, 3)$ satisfy the desired property. Otherwise, $1 + m^2 = (m - 1)(m + 1) + 2$, so if $1 + m^2$ is divisible by $m - 1$, then 2 is also divisible by $m - 1$. But this can only happen if $m - 1 = 1, 2$. Hence $m = 2, 3$ are indeed the only possibilities.
- If $u = 2$, then we want $(2m - 1) \mid (2 + m^2)$. Therefore, $(2m - 1) \mid (8 + 4m^2)$, and

  $$4m^2 + 8 = (2m - 1)(2m + 1) + 9.$$

  Hence $(2m - 1) \mid 9$. This is only possible if $2m - 1 = 1, 3, 9$, or $n = 1, 2, 5$. We see that $(2, 1)$, $(2, 2)$, $(2, 5)$ satisfy the desired property.

- If $u = 3$, then we want $(3m-1) \mid (3+m^2)$. Therefore, $(3m-1) \mid (27+9m^2)$, and

$$9m^2 + 27 = (3m-1)(3m+1) + 28.$$

  Hence $(3m-1) \mid 28$. This is only possible if $3m-1 = 1, 2, 4, 7, 14, 28$, and the only integers $m$ that satisfy one of these equations are $m = 1, 5$. We see that $(3, 1)$, $(3, 5)$ satisfy the desired property.

- If $u = 4$, then we want $(4m-1) \mid (4+m^2)$. Therefore, $(4m-1) \mid (64+16m^2)$, and

$$16m^2 + 64 = (4m-1)(4m+1) + 65.$$

  Hence $(4m-1) \mid 65$. This is only possible if $4m-1 = 1, 5, 13, 65$, and none of these yield integers $m$. Therefore, there are no solutions with $u = 4$.

- If $q = 1$, then the discriminant in (1) is $u^2-4u-4$. This is clearly less than $u^2-4u+4 = (u-2)^2$. Also, if $2u \geq 14$, then $u^2-4u-4 \geq u^2-6u+10 = (u-3)^2+1$. Therefore, if $u \geq 7$, then $u^2-4u-4$ cannot be a perfect square. Checking through $u = 1, 2, \ldots, 6$, we find that $u^2-4u-4$ is only a perfect square for $u = 5$, and then by (1), $m = \frac{5 \pm \sqrt{25-24}}{2} = 2$ or 3. We see that $(5, 2)$ and $(5, 3)$ satisfy the desired property.

- If $q = 2$, then the discriminant in (1) is $4u^2-4u-8$. This is strictly less than $4u^2-4u+1 = (2u-1)^2$. Also, if $4u \geq 13$, then $4u^2-4u-8 \geq 4u^2-8u+5 = (2u-2)^2+1$. Therefore, $4u^2-4u-8$ is not a perfect square for $u \geq 4$. Checking through $u = 1, 2, 3$, we find that $4u^2-4u-8$ is only a perfect square when $u = 2, 3$. If $u = 2$, then by (1), $m = \frac{4 \pm \sqrt{16-16}}{2} = 2$, while if $u = 3$, then by (1), $m = \frac{6 \pm \sqrt{36-20}}{2} = 1$ or 5. We see that $(2, 2)$, $(3, 1)$, and $(3, 5)$ satisfy the desired property.

- If $q = 3$, then the discriminant in (1) is $9u^2-4u-12$. This is strictly less than the perfect square $(3u)^2$. Also, if $2u \geq 14$, then $9u^2-4u-12 \geq 9u^2-6u+2 \geq (3u-1)^2+1$. Therefore, if $u \geq 7$, then $9u^2-4u-12$ cannot be a perfect square. Checking through $u = 1, 2, \ldots, 6$, we find that $9u^2-4u-12$ is only a perfect square for $u = 2$. Then by (1), $m = \frac{6 \pm \sqrt{36-20}}{2} = 1$ or 5. We see that $(2, 1)$ and $(2, 5)$ satisfy the desired property.

Therefore, the only $(u, m)$ such that $u + m^2$ is divisible by $um - 1$ are

$$(1, 2), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 5), (5, 2), (5, 3).$$