1. Suppose that the three trolls have a, b, and c pancakes, respectively.

   We first claim that none of a, b, and c can be prime. If (without loss of generality) a were prime, then a would be divisible by both $\gcd(a, b)$ and $\gcd(a, c)$, which are distinct numbers greater than 1, contradicting the fact that a is prime. Therefore, we may assume that a, b, and c are composite. The first few composite numbers are $4, 6, 8, 9, 10, 12, \ldots$

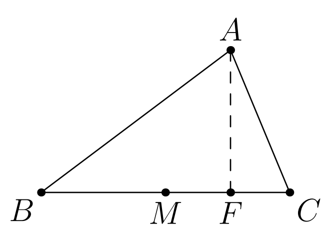
   We next note that if two of the numbers are equal, say $a = b$, then the second condition is violated because $\gcd(a, c) = \gcd(b, c)$. Therefore, the three numbers are distinct.

   If we choose $a = 4$, $b = 6$, and $c = 12$, the three properties are satisfied. We claim that $22 = 4 + 6 + 12$ is the smallest possible sum of a, b, and c.

   If none of the numbers equal 4, then the sum must be greater than or equal to $6+8+9 = 23$, but we can already do better than that. Therefore, in a minimal n, we must have one of the numbers (say a) equal to 4. $\gcd(4, b)$ and $\gcd(4, c)$ are both distinct factors of 4 greater than 1. Without loss of generality, they must be 2 and 4, respectively. So b is an odd multiple of 2 greater than 4, and c is a multiple of 4 greater than 4. So $\gcd(b, c)$ must be an odd multiple of 2, so it is at least 6. Then b, c are at least the smallest two multiples of 6, so the total sum is at least $4 + 6 + 12 = 22$, as desired.

2. We split this into cases based on where the foot F of the alitude from A to BC lies. We may assume that B lies to the left of C on $\overleftrightarrow{BC}$.

   If F lies within the segment BC, then we have the following picture.

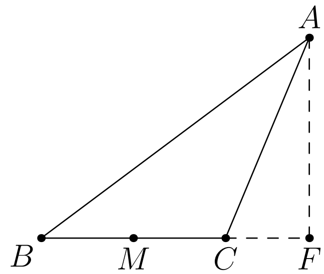


   Using Pythagorean triples, we find that $BF = 16$ and $FC = 5$. Therefore, $BC = 21$, and in particular, $BM = MC = 21/2$ or 10.5. Thus $MF = 21/2 - 5 = 11/2$. By the Pythagorean Theorem,

$$AM = \sqrt{MF^2 + AF^2} = \sqrt{(11/2)^2 + 12^2} = \sqrt{697}/2.$$

   This is achieved by the triangle described above.

   If F lies to the right of C on $\overleftrightarrow{BC}$, then we have the following picture.



Using Pythagorean triples, we find that $BF = 16$ and $CF = 5$. Therefore, $BC = 11$, and $BM = MC = 11/2$. Hence $MF = 11/2 + 5 = 21/2$. By the Pythagorean Theorem,

$$AM = \sqrt{MF^2 + AF^2} = \sqrt{(21/2)^2 + 12^2} = \sqrt{1017}/2.$$

This is also achieved by the described triangle.

The final case is if F lies to the left of B on $\overleftrightarrow{BC}$. As in the previous two cases, we would compute that $BF = 16$ and $CF = 5$, but then this would place C to the left of B on $\overleftrightarrow{BC}$, which contradicts our assumption.

Therefore, the only possible values of AM are $\sqrt{697}/2$ and $\sqrt{1017}/2$.

3. By the Binomial Theorem,

$$(x+y)^n - x^n - y^n = \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1}.$$

Therefore, if $n \geq 2$, then we must have $3 \mid \binom{n}{1}$, which is the same as $3 \mid n$. If $n \geq 4$, then we must have $3 \mid \binom{n}{3}$, or rather,

$$3 \mid \frac{n(n-1)(n-2)}{6}.$$

As n is divisible by 3, we know that $(n-1)$ and $(n-2)$ are not divisible by 3. Therefore, the only factors of 3 in the numerator can come from n. We need one factor of three to cancel with the factor of 3 in the denominator, and one factor to make the integer divisible by 3. Therefore, $9 \mid n$. If $n \geq 10$, then we must have $3 \mid \binom{n}{9}$, which (by a similar argument) implies $27 \mid n$. Thus the only numbers n to check are $n = 3, 9$. In fact, both of them satisfy the desired property.

Alternate Solution: If the gcd of the coefficients is divisible by 3, then the sum of the coefficients is divisible by 3, hence $(1+1)^n - (1+1) = 2^n - 2$ is divisible by 3. But as $2^2 \equiv 1 \pmod 3$, we see that

$$2^n \equiv \begin{cases} 1 \pmod 3 & \text{if } n \text{ is even} \\ 2 \pmod 3 & \text{if } n \text{ is odd.} \end{cases}$$

Hence $2^n - 2$ is divisible by 3 if and only if n is odd. Also, the coefficient of $x^{n-1}y$ is n, so n must be divisible by 3. Hence n must be congruent to $3 \pmod 6$, leaving $n = 3, 9, 15$. We check that $n = 3, 9$ satisfy the property, and for $n = 15$, we find that $\binom{15}{3}$ is not divisible by 3, so 3 and 9 are the only ones that work.

4. We claim they can do it in a minimum of 16 hours.

Firstly, we show this is possible in the following way. For the first 6 hours, Balthazar takes the bike and goes 60 km. Then, Balthazar drops off the bike and walks for the remaining 10 hours, in which time he goes $4 \cdot 10 = 40$ km and thus reaches the destination 100 km away. Meanwhile, Anastasia walks for the first 12 hours, in which time she walks $12 \cdot 5 = 60$ km and arrives at the bike Balthazar dropped off. She then bikes for the remaining 4 hours, in which time she covers the remaining 40 km.

Now we show that no matter how they do this, they will take at least 16 hours. We may assume that Anastasia and Balthazar only stop, turn around, switch from biking to walking, or vice versa at a finite number of locations; fix these locations and suppose that the paths between locations are represented by line segments $1, 2, 3, \ldots, n$, with the length of line segment i being a_i, and with $a_1 + a_2 + a_3 + \cdots + a_n = 100$ km.

Fix a specific line segment i. Let x be the number of times a person bikes forward across it, x' be the number of times a person bikes backward across it, y be the number of times a person walks forward across it, and y' be the number of times a person walks backward across it. Since both Anastasia and Balthazar eventually get to the other side, $x + y - x' - y' = 2$. Since the bike must end up one one side or the other of the segment, $x - x' = 0$ or 1. Thus $y - y' = 1$ or 2, and in particular $y \geq 1$. Thus either Anastasia walks forward across this line segment at some point, or Balthazar does (or both).

Let A be the total length Anastasia walks forward, and let B be the total length Balthazar walks forward. Since either Anastasia or Balthazar walks across each segment, $A + B \geq 100$ km. The total time spent walking by Anastasia is at least $A/5$ and the total time spent walking by Balthazar is at least $B/4$. Additionally, Anastasia spends at least $(100 - A)/10$ hours biking and Balthazar spends at least $(100 - B)/10$ hours biking. Thus if the total time it takes them is T, we have

$$T \geq \frac{A}{5} + \frac{100 - A}{10} = 10 + \frac{1}{10}A$$
$$T \geq \frac{B}{4} + \frac{100 - B}{10} = 10 + \frac{3}{20}B.$$

Adding 30 times the first inequality to 20 times the second inequality, we find

$$30T + 20T \geq 300 + 200 + 3A + 3B = 500 + 3(A + B).$$

Employing our observation earlier that $A + B \geq 100$, we find that

$$50T \geq 500 + 3(100) = 800,$$

so

$$T \geq \frac{800}{50} = 16 \text{ hours.}$$

5. We fill the grid by placing into the grid all the factors of 5, 2, and -1 in turn.

   The factors of 5 can be distributed in the grid in 6 ways. (There are 3 ways to place a 5 in the first column, 2 remaining possible slots in the second column, and one remaining possible slot in the third column, so $3 \cdot 2 \cdot 1 = 6$.)

   We next show that the factors of 2 can be distributed in the grid in 21 ways.

   - **Case 1:** The grid contains a 4 (that is, we put two factors of 2 in the same cell somewhere)
     - **Subcase 1.1:** The grid contains another 4. This second 4 must be placed in a different row and column, and clearly, another 4 must be placed. So this subcase is essentially the same as the 5's above, and it can be done in 6 ways.
     - **Subcase 1.2:** The grid does not contain another 4. Then the entries in the same row/column as the 4 must be 1's, and the other entries must all be 2's. Everything is determined by the location of first 4, so this can be done in 9 ways.
   - **Case 2:** The grid contains no 4's

     Then each row/column contains two 2's and one 1. Therefore, each row/column contains exactlyr one 1. Therefore, this can be done in exactly the same method as the 5's above, for a total of 6 ways.

   Therefore, we can place the 2's in a total of $6 + 9 + 6 = 21$ ways.

   The factors of -1 can be distributed in the grid in 16 ways. (Any particular distribution of signs in the grid is determined by the sign distribution in the upper-left-hand 2×2 grid of boxes; the rest of the signs are determined from there, and regardless of those four values, the rest of the grid works out so that the product of any row or column is positive. So $2^4 = 16$.)

   The total number of ways of filling the grid is $6 \times 21 \times 16 = 2016$.

6. First, suppose that the plane that contains the upper circular base of the cylinder intersects OA, OB, and OC in the points A', B', and C', respectively. We claim that if $OABC$ achieves the minimum possible height, then the upper circular base of the cylinder is the inscribed circle of $\triangle A'B'C'$.

   If not, then we may suppose without loss of generality that $A'B'$ is not tangent to the circular base. Then, within the triangle $\triangle A'B'C'$ we can draw line $A''B''$ parallel to side $A'B'$, such that A'' is on $A'C'$ and B'' is on line $B'C'$ and $A''B''$ is tangent to the circular base. As $A''B''$ is parallel to $A'B'$, which in turn is parallel to AB, then A, B, A'', B'' are coplanar.

Let O' be the intersection of plane $ABA''B''$ with line OC. The new pyramid $O'ABC$ contains the cylinder, and it has strictly smaller height than pyramid $OABC$. It also has a higher number of faces tangent to the upper circular base of the cylinder than pyramid $OABC$. By applying this process on each side of the triangle ABC, we obtain a pyramid with strictly smaller height and all lateral faces tangent to the upper circular base. Therefore, given a triangular base ABC and apex O, we can always find a pyramid with smaller height and all three lateral faces tangent to the upper circular base of the cylinder. Hence we may assume that the upper circular base of the cylinder is the incircle of $\triangle A'B'C'$.

Now as face ABC is parallel to face $A'B'C'$, we know that pyramids $OABC$ and $OA'B'C'$ are similar. Therefore, the similarity ratio tells us that if r is the inradius of $\triangle ABC$ and if h is the height from O of pyramid $OABC$, then

$$\frac{\text{inradius}(ABC)}{\text{inradius}(A'B'C')} = \frac{\text{height}(OABC)}{\text{height}(OA'B'C')}$$
$$\frac{r}{4} = \frac{h}{h-10}.$$

Solving for h, we find that

$$h = \frac{10r}{r-4} = 10 + \frac{40}{r-4}.$$

Clearly, $r > 4$, and by the above equation, as r increases, the value of h must decrease. So the problem simplifies to finding the maximum possible inradius of a triangle with perimeter 84.

In $\triangle ABC$, let D, E, and F be the points of tangency of the incircle with sides BC, CA, and AB, respectively. Also, let $x = AE = AF$, $y = BD = BF$, $z = CD = CE$. Then the perimeter of $\triangle ABC$ is

$$2(x+y+z) = 84.$$

Hence the semiperimeter, s, is given by $s = x+y+z = 42$. Note that $x = s-a$, $y = s-b$, and $z = s-c$. Also, the area of $\triangle ABC$ (by the inradius formula and Heron's formula) is

$$42r = \sqrt{42xyz}.$$

Hence

$$r = \sqrt{\frac{xyz}{42}}.$$

By the AM-GM inequality,

$$xyz \leq \frac{(x+y+z)^3}{27} = 14^3,$$

with equality if and only if $x = y = z$, i.e. $\triangle ABC$ is equilateral. Hence

$$r = \sqrt{\frac{xyz}{42}} \leq \frac{14}{\sqrt{3}}.$$

This leads to a height of $h = \frac{490+140\sqrt{3}}{37}$, achieved when we have base ABC with side lengths $AB = BC = CA = 28$.