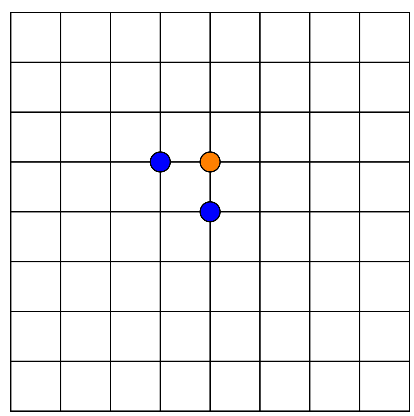
The Second Annual Utah Math Olympiad–Solutions

1. First, note that both Todd and Allison place exactly one blue stone in each of their turns. Also, it will take Todd at least eight moves to get from the center to a corner. Therefore, as Todd would go both first and last, it would take at least $8 + 7 = 15$ turns of Todd and Allison combined before Todd could move to a corner square. Therefore, at least 15 blue stones are placed on the grid.

   We claim that Allison can always force Todd to move to a corner square in his eighth turn. Without loss of generality, we may assume that Todd's first turn is to move north. Allison decides that she wants to force Todd to the northeast corner, namely corner D. In order to do this, she plays immediately to the West of Todd, blocking any westward motion. So after Todd and Allisons' first turns, we may assume that the board looks like the diagram below.

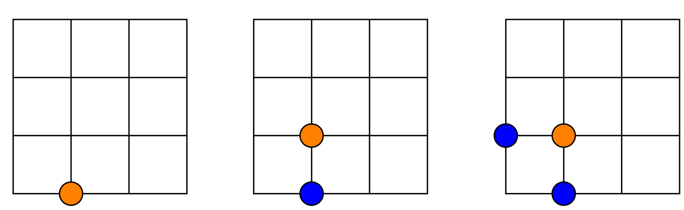


   Note that Todd can only move north or west. After this point, Allison executes the following strategy based on Todd's turn.

   (i) If Todd moves north, then Allison plays directly to his west.

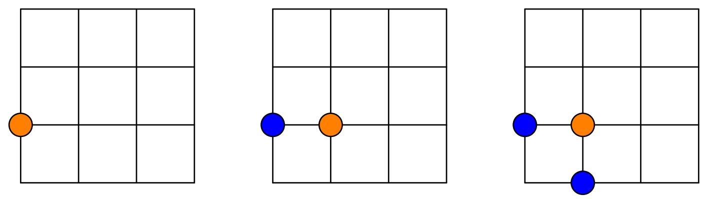
   (ii) If Todd moves east, then Allison plays directly to his south.

   In order for this strategy to make sense, we must show that after Allison moves, then Todd can only move to the north or to the west. Let's look at a very general picture. If Todd moves north, then he leaves a blue stone behind him, and Allison plays to his west. This leaves the following arrangement of the board.



Therefore, in this case, Todd can only move north or east.

In the case that Todd moves to the east, then Allison moves directly to his south, so we obtain the following picture.



Once again, it is clear that Todd can only move to the north or to the east.

Therefore, after Allison moves according to her strategy, Todd can only move in the north or east direction. But he can only make 4 north moves before reaching the edge of the board, and 4 east moves before reaching the edge of the board. Therefore, according to the described strategy of Allison, she can force Todd to reach a corner in 8 steps. This is also the minimal number of steps that it would take Todd to reach a corner, so we conclude that this is Allison's optimal strategy. Therefore, as stated earlier, we conclude that at the end of the game, there will be exactly 15 blue stones on the board.

2. We present a solution that works for both parts (a) and (b).

   There are no solutions to the equations in question. Consider the equations mod 8. They become the same congruence.

   $$x^2 + y^2 \equiv 6 \pmod{8}.$$

   If the equations had a solution, then this congruence would also have a solution, so it suffices to show that this congruence has no solutions. The only perfect squares mod 8 are 0, 1, and 4. Therefore, the only possibilities for the sums of squares are $0, 1, 2, 4, 5 \pmod{8}$. Therefore, the sum of two squares is never congruent to $6 \pmod{8}$. Hence the equations have no solutions.

   We can also approach this problem without directly using the language of modular arithmetic. Suppose that the first equation has some solution (x, y). If x is even, then x^2 will be even, and if x is odd, then x^2 will be odd. Therefore, because the sum of an even number and an odd number is odd, we know that either x and y are both even, or else they are both odd. If x and y are both even, let $x = 2x'$ and $y = 2y'$ for integers x' and y'. Then the first equation becomes

   $$4(x')^2 + 4(y')^2 = 2014.$$

   But $2014 = 2 \cdot 1007$, so it is not divisible by 4, so this case is impossible. Therefore, x and y must both be odd. Therefore, we can represent them by $x = 2x' - 1$ and $y = 2y' - 1$ for integers x' and y'. Then the first equation becomes

   $$(4(x')^2 - 4x' + 1) + 4(y')^2 - 4y' + 1) = 2014.$$

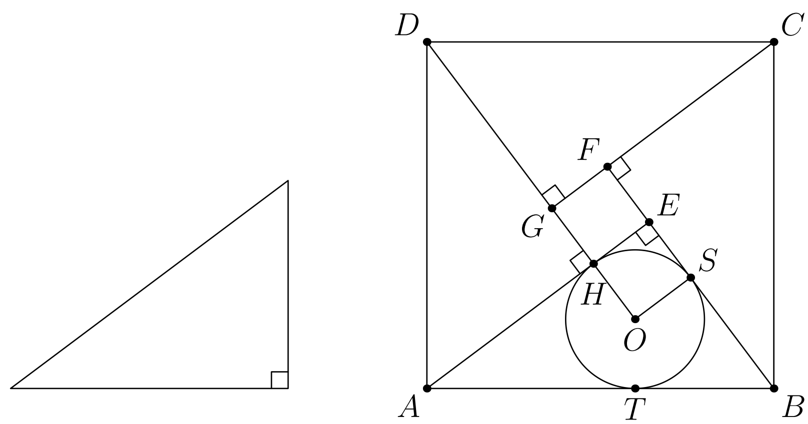
$$4((x')^2 - x' + (y')^2 - y') = 2012$$

$$(x')^2 - x' + (y')^2 - y' = 503.$$

Now $(x')^2 - x = x'(x'-1)$ is the product of two consecutive integers, and so one of these consecutive integers must be even. Therefore, the product must be even. The same applies to $(y')^2 - y'$, hence their sum must be even. But the sum is 503, which is odd, so we have a contradiction. Therefore, the first equation has no solutions.

The same argument can be adapted for part (b) of the problem.

3. Let the legs of the right triangle be a and b, with $a \leq b$. Suppose that the right triangle satisfies the given properties. Then we have the following picture.



We know that $AE = b$, $BE = a$, and $AH = a$, therefore $EH = b - a$. We know that $EH = ES$, as they are tangents to circle O from the same point (or by power of a point), and hence $ES = b - a$. But as $BE = a$, this implies that $BS = BE - ES = a - (b - a) = 2a - b$. We also find that $TB = BS = 2a - b$, as they are tangents to circle O from the same point.

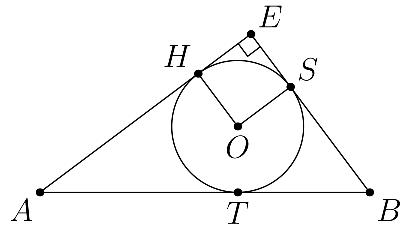
Similarly, as $AH = a$, we know that $AT = AH = a$. Therefore, $AB = AT + TB = a + (2a - b) = 3a - b$. Therefore, by the Pythagorean Theorem applied to triangle ABC, we know that

$$a^2 + b^2 = (3a - b)^2 = 9a^2 - 6ab + b^2$$

$$0 = 8a^2 - 6ab.$$

Therefore, $a(4a - 3b) = 0$. So either $a = 0$, which leads to degenerate triangles, or else $4a = 3b$. Therefore, we know that a is a multiple of 3, say $a = 3a'$ and b is a multiple of 4, say $b = 4b'$. Plugging this into the equation tells us that $12a' = 12b'$, or $a' = b'$. Therefore, all such triangles have legs of lengths $3n$ and $4n$, and a hypotenuse of length $5n$.

Now we must check that all such right triangles satisfy the desired property. Suppose we are given a right triangle ABE with side lengths $AB = 5n$, $BE = 3n$, and $EA = 4n$, where n is a positive integer, and points are labeled below.



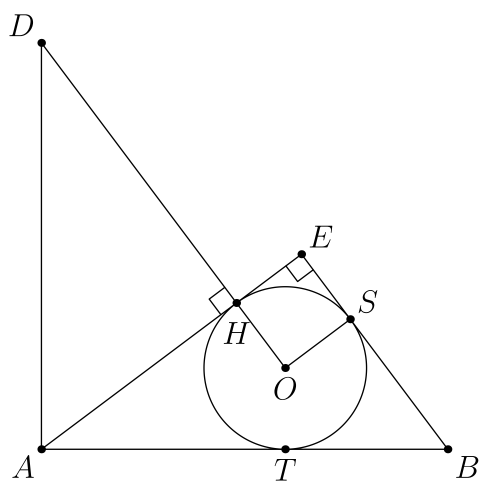
Then the perimeter is given by

$$12n = EH + HA + AT + TB + BS + SE = 2 \cdot EH + 2 \cdot AT + 2 \cdot TB$$

by the fact that $SE = EH$, $HA = AT$, and $TB = BS$. Therefore,

$$2(EH + AT + TB) = 2(EH + AB) = 2(EH + 5n) = 12n,$$

so we conclude that $EH = SE = n$. We can use a similar method to show that $HA = AT = 3n$ and $TB = BS = 2n$. Therefore, $EB = HA$. Now we arrange two of the triangles into a part of the square formation.



As $AH = 3n$, angle AHD is right, and $OH \perp AE$, we know that D, H, and O are collinear. Because DH contains the side of a smaller square, we know that O lies on the extension of the smaller square. By symmetry, the same must apply for the other three triangles. Therefore, all triangles with sides lengths $3n$, $4n$, and $5n$, where n is a positive integer do indeed satisfy the desired property. Thus they describe all such triangles.

4. If Joel starts out with a list whose length is not 2, and eventually encounters a list (m, n) of length 2, he might as well have started out with the list (m, n) (because we only care about what lists appear an infinite number of times). On the other hand if he never encounters a list (m, n) of length 2, we can safely ignore this case as no lists of length 2 will appear infintely often. Therefore, we may assume that Joel starts out with a list of length 2.

   We claim that for any $m, n \geq 0$, $(m, n) \to \cdots$ eventually arrives at one of the following two repeating patterns:

$$(1,2) \to (0,1,1) \to (1,2) \to \cdots \tag{$*$}$$
$$(2,2) \to (0,0,2) \to (2,0,1) \to (1,1,1) \to (0,3) \to (1,0,0,1) \to (2,2) \to \cdots \tag{$**$}$$

   Clearly, this is true for the lists $(m, n) = (1, 2), (2, 2)$, and $(0, 3)$. Next, consider the case where $(m, n) = (k, 2)$ or $(2, k)$, for $k > 2$:

$$\begin{matrix}(k,2)\to\\(2,k)\to\end{matrix}\ (0,0,1,\underbrace{0,0,\ldots,0}_{k-3\text{ zeros}},1) \to (k-1,2)$$

   Then $(k-1, 2)$ will eventually become $(k-2, 2)$, and so on, until we arrive at $(2, 2)$, which will repeat pattern $(**)$.

   Next, consider all other cases where $m = 2$ or $n = 2$ not yet covered:

$$(2,1) \to (0,1,1) \to (1,2)$$
$$\begin{matrix}(2,0)\to\\(0,2)\to\end{matrix}\ (1,0,1) \to (1,2)$$

   and $(1, 2)$ repeats pattern $(*)$.

   We now know that if $m = 2$ or $n = 2$, (m, n) eventually reduces to one of the two repeating patterns $(**)$ or $(*)$. Now more generally, consider the case $(m, n) = (a, b)$ or (b, a), where $0 \leq a < b$. Then

$$\begin{matrix}(b,a)\to\\(a,b)\to\end{matrix}\ (\underbrace{0,0,\ldots,0}_{a\text{ zeros}},1,\underbrace{0,0,\ldots,0}_{b-a-1\text{ zeros}},1) \to (b-1,2)$$

   which we have already shown eventually repeats one of the two patterns.

   The only case remaining is $(m, n) = (k, k)$ for some $k \neq 2$. For small k, this is:

$$(0,0) \to (2) \to (0,0,1) \to (2,1)$$
$$(1,1) \to (0,2)$$
$$(3,3) \to (0,0,0,2) \to (3,0,1) \to (1,1,0,1) \to (1,3)$$

all three cases $(2,1), (0,2), (1,3)$ have already been covered. Now when $k \geq 4$,

$$(k,k) \to (\underbrace{0,0,\ldots,0}_{k \text{ zeros}},2) \to (k,0,1) \to (1,1,\underbrace{0,0,\ldots,0}_{k-2 \text{ zeros}},1) \to (k-2,3)$$

At this point either $k-2=3$, or $k-2 \neq 3$; we dealt with both of these cases before.

The only pairs (m,n) which Joel could end up writing infinitely many times are those which appear in one of the two repeating patterns:

$$\boxed{(1,2),(2,2),(0,3)}.$$

5. Ignore the first equation for now. From the second equation $x^2+y^2+z^2 = 4x\sqrt{yz} - 2yz$, deduce

$$(x^2 - 4x\sqrt{yz} + 4yz) + (y^2 - 2yz + z^2) = 0$$
$$(x - 2\sqrt{yz})^2 + (y-z)^2 = 0$$

Since x, y, z are real, and as the sum of two squares can only be 0 when both squares are 0 (as squares are nonnegative), this implies $y = z$, and

$$x - 2\sqrt{yz} = 0 \implies x = 2\sqrt{zz} = 2|z| = 2z$$

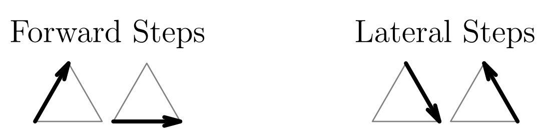
Now we recall the first equation.

$$xyz = 1 \implies (2z)(z)(z) = 1 \implies z^3 = \frac{1}{2} \implies z = \frac{1}{\sqrt[3]{2}}$$

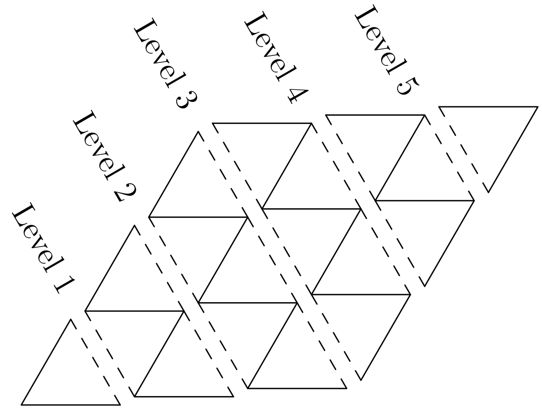
Therefore,

$$\boxed{x = \frac{2}{\sqrt[3]{2}}\ ,\ y = \frac{1}{\sqrt[3]{2}}\ ,\ z = \frac{1}{\sqrt[3]{2}}}.$$

6. First, we place the steps into two different categories as labeled below. The first category is called the *forward* steps, while the second category is called the *lateral* steps.



In a path from A to B, there are exactly $2n$ forward steps. To see this, split the diagram at what we will term *levels* as shown below. Each level consists of the segments along which we can move laterally without moving forward.



A path consists of a step from Level 0 to level 1 (the first forward step), followed by some sequence of steps in level 1, followed by a step from level 1 to level 2 (the second forward step), followed by some sequence of steps in level 2, and so on. Note that due to the restrictions on steps, once we reach a level, we cannot return to a previous level. Therefore, we can only make one forward step between levels, so we will make exactly $2n$ forward steps.

Given a path, we split it up into its forward steps and its lateral steps. We claim that the lateral steps are completely determined (or rather, forced) by our choice of forward steps. If we know the forward steps of a path, then the nth forward step will start at level $n-1$ and end at a point on level n. The $(n+1)$th forward step will start at another point on level n. But then there is a unique path in level n between these points without retracings because we can only move in one direction in a level. Therefore, every path is uniquely determined by its forward steps.

Also, given a sequence of $2n$ forward steps with one forward step in each gap between levels, we can fill in lateral steps uniquely as well, and hence there is a unique path associated to each sequence of $2n$ forward steps. This establishes a one-to-one correspondence between paths and sequences of $2n$ forward steps, each starting at a different level.

Therefore, the number of paths is equal to the number of ways to select $2n$ forward steps, with exactly one forward step between two levels. Referring to the above diagram, for the first forward step, we have 2 choices, for the second forward step we have 4 choices, $\ldots$, for the nth forward step we have $2n$ choices, for the $(n+1)$th forward step we have $2n$ choices, for the $(n+2)$th forward step we have $2(n-1)$ choices, $\ldots$, for the $(2n)$th forward step we have 2 choices. Therefore, the total number of paths is

$$(2 \cdot 4 \cdot 6 \cdots (2n))((2n) \cdot (2n-2) \cdots 2) = (2 \cdot 4 \cdot 6 \cdots (2n))^2.$$

This is always a perfect square, so the result clearly follows.