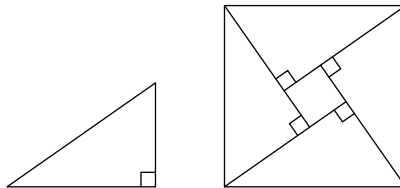# The Second Annual Utah Math Olympiad
March 22, 2014

1. Todd and Allison are playing a game on the grid shown below. At the beginning, an orange stone is placed in the center intersection on the grid. They take turns, with Todd going first. In each of Todd's turns, he must move the orange stone from its current position to a horizontally or vertically adjacent intersection that is not occupied by a blue stone, and then he places a blue stone in the orange stone's previous spot. In each of Allison's turns, she places a blue stone on exactly one unoccupied intersection. Todd loses the game when he is forced to move into one of the corner intersections, labeled by A, B, C, and D in the diagram below. Allison loses if Todd can't move.

   Allison tries to force Todd to lose in as few as turns as possible, and Todd tries to survive as long as possible. If both of them play as best they can, how many blue stones will be on the board at the end of the game? (You may assume that Todd always loses.)
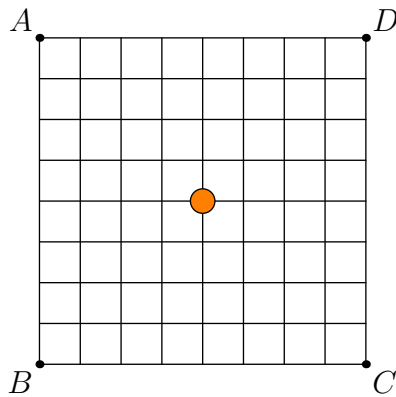
2. (a) Find all positive integers x and y that satisfy $x^2 + y^2 = 2014$, or prove that there are no solutions.

   (b) Find all positive integers x and y that satisfy $x^2 + y^2 = 3222014$, or prove that there are no solutions.

3. Completely describe the set of all right triangles with positive integer-valued legs such that when four copies of the triangle are arranged in square formation shown below, the incenters of the four triangles lie on the extensions of the sides of the smaller square. (Note: the *incenter* of a triangle is the center of the circle inscribed in that triangle.)

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4. Joel is playing with ordered lists of integers in the following way. He starts out with an ordered list of nonnegative integers. Then, he counts the number of 0's, 1's, 2's, and so on in the list, writing the counts out as a new list. He stops counting when he has counted everything in the previous list. Then he takes the second list and applies the same process to get a third list. He repeats this process indefinitely.

   For example, he could start out with the ordered list $(0, 0, 0, 2)$. He counts three 0's, zero 1's, and one 2, and then stops counting, so the second list is $(3, 0, 1)$. In the second list, he counts one 0, one 1, zero 2's, and one 3, so the third list is $(1, 1, 0, 1)$. Then he counts one 0 and three 1's, so the fourth list is $(1, 3)$. Here are the first few lists he writes down:

$$(0, 0, 0, 2) \longrightarrow (3, 0, 1) \longrightarrow (1, 1, 0, 1) \longrightarrow (1, 3) \longrightarrow \cdots$$

   If instead he started with $(0, 0)$, he would write down:

$$(0, 0) \longrightarrow (2) \longrightarrow (0, 0, 1) \longrightarrow (2, 1) \longrightarrow \cdots$$

   If Joel starts out with an arbitrary list of nonnegative integers and then continues this process, there are certain lists (m, n) of length two that he might end up writing an infinite number of times. Find all such pairs (m, n).

5. Find all positive real numbers x, y, and z that satisfy both of the following equations.

$$xyz = 1$$

$$x^2 + y^2 + z^2 = 4x\sqrt{yz} - 2yz.$$

6. Draw n rows of $2n$ equilateral triangles each, stacked on top of each other in a diamond shape, as shown below when $n = 3$. Set point A as the southwest corner and point B as the northeast corner. A step consists of moving from one point to an adjacent point along a drawn line segment, in one of the four legal directions indicated. A path is a series of steps, starting at A and ending at B, such that no line segment is used twice. One path is drawn below. Prove that for every positive integer n, the number of distinct paths is a perfect square. (Note: A perfect square is a number of the form k^2, where k is an integer).

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