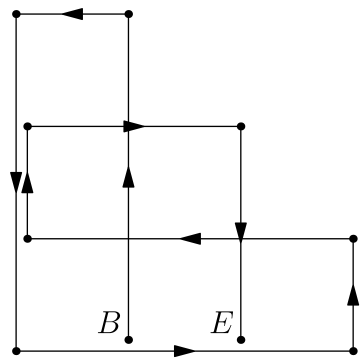
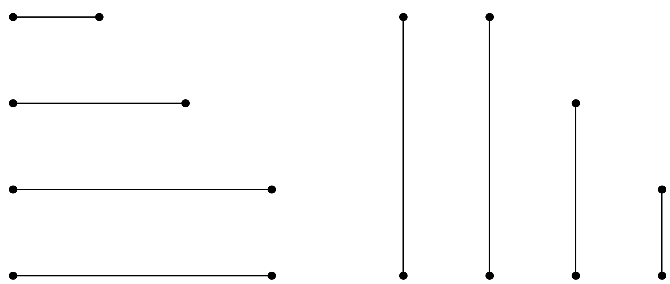
The First Annual Utah Math Olympiad–Solutions

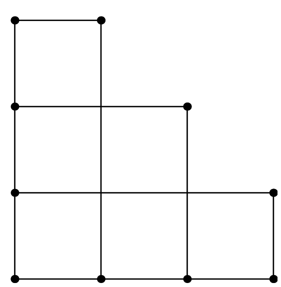
1. (a) The path below that starts at B and ends at E and travels along the segments in the direction of the arrows retraces the diagram in exactly nine segments without lifting the pencil from the page.



(b) Suppose we dissect the diagram into two pieces as shown.



Note that we have eight distinct segments, so if we had a retracing that used exactly eight segments, then these would have to be our segments (connected in some order). The dots shown above are the only locations where we can change direction. If we are counting the number of dots in a valid path, we will count one for the starting point, seven for the changes in direction, and one for the ending point, for a total of nine dots. If we overlay the two diagrams above, however, we count ten distinct dots that we must visit.



This is a contradiction, so we cannot retrace the diagram using eight segments.

2. If A is the area of a square, then we claim that Alice has a winning strategy if and only if $1 < A < 2$ or $2 < A \leq 4$.

   First, note that If $1 < A < 2$, then Alice will win because if she cuts the paper in half, then Carl will receive a sheet of paper of area less than 1 and lose. If $A = 2$, then Alice can cut off any area up to half, so the remaining area will be greater than or equal to 1, but less than 2. Thus Carl will not lose on his first turn. He then cuts off half, giving Alice a sheet of area less than 1. In this case, Carl wins. Therefore, if a player receives 2, the other player has a winning strategy. Therefore if $2 < A \leq 4$, then Alice will cut the paper into sheets of size 2 and $A - 2$. Then Carl will receive the sheet of size 2, and Alice has a winning strategy. We claim that for all other areas above 4, the optimal strategy leads the game to go on infinitely. First, consider what happens when a player receives a sheet of paper with area $4 < A \leq 8$. Then when they make their cut, the paper that they pass will have area greater than 2 and less than 8. But if they give the other player a piece of paper of area $2 < A \leq 4$, then as noted above, that player will have a winning strategy. Therefore, in order to not lose, the player will make a cut that leaves the other player with an area greater than 4. This process would theoretically go on infinitely if both players pursue optimal strategies.

3. **Solution 1**

   We claim that the only possibilities for x are 38, 538, 462, and 962.

   Let $x = 100a + 10b + c$, with $a, b, c \in \{0, 1, 2, ..., 9\}$. We are looking for when the last three digits of x^2 are ddd, with $d \in \{1, 2, ..., 9\}$. Now the problem becomes finding when $(100a+10b+c)^2 \equiv 111d$ (mod 1000). Expanding out $(100a + 10b + c)^2$ this becomes

$$200ac + 100b^2 + 20bc + c^2 \equiv 100d + 10d + d \pmod{1000} \tag{1}$$

   Since 20 divides 1000, we can take this equation modulo 20 to simplify things. Doing this, it becomes $c^2 \equiv 11d$ (mod 20). Now consider all possible cases for c and d.

c	c^2 (mod 20)
0	0
1	1
2	4
3	9
4	16
5	$25 \equiv 5$
6	$36 \equiv 16$
7	$49 \equiv 9$
8	$64 \equiv 4$
9	$81 \equiv 1$

d	$11d$ (mod 20)
1	11
2	$22 \equiv 2$
3	$33 \equiv 13$
4	$44 \equiv 4$
5	$55 \equiv 15$
6	$66 \equiv 6$
7	$77 \equiv 17$
8	$88 \equiv 8$
9	$99 \equiv 19$

   As you can see, the only possibility for when $c^2 \equiv 11d$ (mod 20) is if $c^2 \equiv 11d \equiv 4$, which implies $c = 2$ or 8 and $d = 4$. We split into the cases $c = 2$ and $c = 8$.

Case 1: $c = 2$

Return to equation (1) and plug in the new values for c and d:

$$400a + 100b^2 + 40b + 4 \equiv 444 \pmod{1000}$$

$$400a + 100b^2 + 40b \equiv 440 \pmod{1000}$$

Now divide by the common factor of 20 to get:

$$20a + 5b^2 + 2b \equiv 22 \pmod{50} \tag{2}$$

This implies $5b^2 + 2b \equiv 2 \pmod{10}$. Modulo 2 we find $5b^2 \equiv 0$, so b must be even; modulo 5 we must have $2b \equiv 2$ so $b \equiv 1 \pmod 5$. By Chinese remainder theorem, $b \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 5$ imply $b \equiv 6 \pmod{10}$, so $b = 6$.

Now returning to (2) we get

$$20a + 5b^2 + 2b = 20a + 5 \cdot 36 + 2 \cdot 6 \equiv 22 \pmod{50}$$

$$\Rightarrow 20a + 20 \equiv 0 \pmod{50}$$

$$\Rightarrow 2a + 2 \equiv 0 \pmod 5$$

$$\Rightarrow a \equiv 4 \pmod 5 \Rightarrow a = 4 \text{ or } a = 9$$

So we have the solutions 462 and 962, which we check both work.

Case 2: $c = 8$

Return to equation (1) and plug in the new values for c and d:

$$1600a + 100b^2 + 160b + 64 \equiv 444 \pmod{1000}$$

$$1600a + 100b^2 + 160b \equiv 380 \pmod{1000}$$

Now divide by the common factor of 20 to get:

$$80a + 5b^2 + 8b \equiv 19 \pmod{50} \tag{3}$$

This implies $5b^2 + 8b \equiv 9 \pmod{10}$. Modulo 2 we find $5b^2 \equiv 1$, so b must be odd; modulo 5 we must have $3b \equiv 4$ so $b \equiv 3 \pmod 5$. By Chinese remainder theorem, $b \equiv 1 \pmod 2$ and $b \equiv 3 \pmod 5$ imply $b \equiv 3 \pmod{10}$, so $b = 3$.

Now returning to (3) we get

$$80a + 5b^2 + 8b = 80a + 5 \cdot 9 + 8 \cdot 3 \equiv 19 \pmod{50}$$

$$\Rightarrow 80a \equiv 0 \pmod{50}$$

$$\Rightarrow 3a \equiv 0 \pmod 5$$

$$\Rightarrow a \equiv 0 \pmod 5 \Rightarrow a = 0 \text{ or } a = 5$$

So we have the solutions 38 and 538, which we check both work.

Solution 2

We are attempting to solve $x^2 \equiv 111 \cdot y \pmod{1000}$, for $1 \leq y \leq 9$. By $\pmod{10}$ considerations, we can immediately restrict to the cases $111, 444, 555, 666$, and 999. Now, as a consequence of the Chinese Remainder Theorem, the residue of $n \pmod{1000}$ is uniquely determined by the residue of $n \pmod 8$ and $\pmod{125}$. The only perfect squares $\pmod 8$ are $0, 1$, and 4. However, $111 \equiv 7 \pmod 8$, $444 \equiv 4 \pmod 8$, $555 \equiv 3 \pmod 8$, $666 \equiv 2 \pmod 8$, and $999 \equiv \pmod 7$, we can restrict to the case $y = 4$.

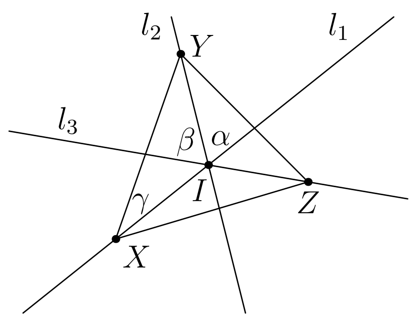
Now, $444 \equiv 4 \pmod 8$ and $444 \equiv 69 \pmod{125}$, so we must solve $x_1^2 \equiv 4 \pmod 8$ and $x_2^2 \equiv 69 \pmod{125}$. Certainly, $x_1 \equiv 2, 6 \pmod 8$, i.e. $x_1 \equiv 2 \pmod 4$. We solve the second equation first $\pmod 5$, then $\pmod{25}$, then $\pmod{125}$:

$x_2^2 \equiv 4 \pmod 5$, so $x_2 \equiv 2, 3 \pmod 5$.

$(5x_2' \pm 2)^2 \equiv 19 \pmod{25}$, so $x_2' \equiv \pm 2 \pmod 5$, and $x_2 \equiv \pm 12 \pmod{25}$. $(5x_2'' \pm 12)^2 \equiv 69 \pmod{125}$, so $x_2'' \equiv \pm 3 \pmod 5$, and $x_2 \equiv \pm 87 \pmod{125} = \pm 38 \pmod{125}$.

So then we have to simply find all x with $x \equiv 2 \pmod 4$, and $x \equiv \pm 38 \pmod{125}$. This gives 38, $38 + 500 = 538$, $87 + 375 = 462$, and $87 + 875 = 962$. Thus the only possibilities are $38, 462, 538$, and 962.

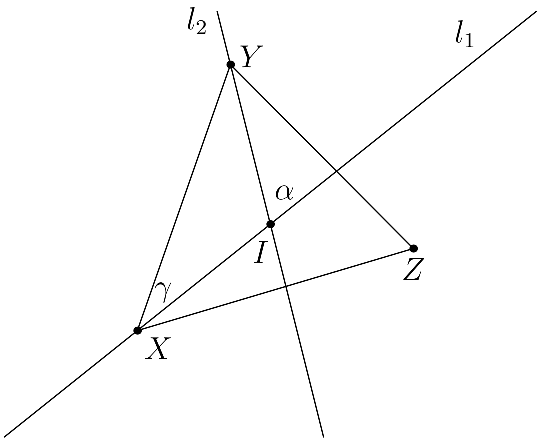
4. We claim that it is necessary and sufficient that $\beta + \gamma = \frac{\pi}{2}$. To prove that this condition is necessary, we do some angle chasing. Suppose that such a triangle XYZ exists with Z on l_3 as shown.

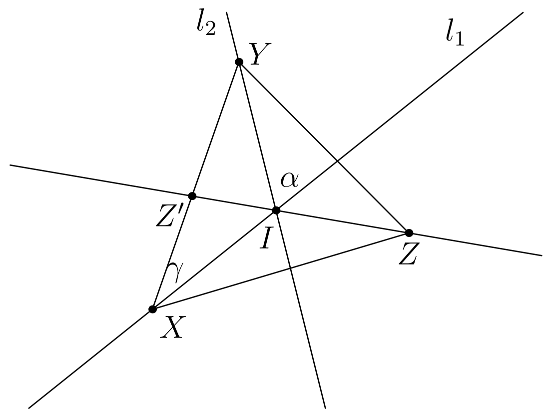


Now $\angle YIX = \pi - \alpha$, so by triangle XYI, we know that $\angle XYI = \pi - (\pi - \alpha) - \gamma = \alpha - \gamma$. By our assumption that the lines are angle bisectors, we have $\angle IYZ = \alpha - \gamma$ as well. Similarly, $\angle IXZ = \gamma$. Now $\angle YIZ = \pi - \beta$, so using triangle IYZ, we have $\angle YZI = \pi - (\pi - \beta) - (\alpha - \gamma) = \beta - \alpha + \gamma$. Using vertical angles, we know that $\angle XIZ = \alpha + \beta$. Thus by triangle XIZ, we know that $\angle IZX = \pi - (\alpha + \beta) - (\gamma) = \pi - \alpha - \beta - \gamma$. As l_3 bisects $\angle XZY$, we know that $\angle XZI = \angle YZI$, or rather

$\pi - \alpha - \beta - \gamma = \beta - \alpha + \gamma$. Simplifying this, we get $\frac{\pi}{2} = \beta + \gamma$. Therefore, for such a triangle to exist, it is necessary that $\beta + \gamma = \frac{\pi}{2}$.

We claim that this condition is also sufficient. Suppose $\beta + \gamma = \frac{\pi}{2}$. First, let's ignore l_3 for the moment as shown below, to the left. From points X and Y, we draw lines at angles γ and $\alpha - \gamma$ respectively. If we call their intersection point Z, then l_1 and l_2 are angle bisectors of $\triangle XYZ$ as shown below, to the right. Then the line passing through I and Z must also be an angle bisector of $\triangle XYZ$ by concurrency of angle bisectors. We claim that line IZ is in fact line l_3, which would tell us that this condition is sufficient.





We proceed again by angle chasing. Once again, $\angle YIX = \pi - \alpha$, so by triangle XYI, we know that $\angle XYI = \pi - (\pi - \alpha) - \gamma = \alpha - \gamma$. As the three lines are all angle bisectors of triangle XYZ, we know that $\angle YZX = \pi - 2(\gamma) - 2(\alpha - \gamma) = \pi - 2\alpha$. Thus $\angle IZY = \angle IZX = \frac{\pi}{2} - \alpha$. By triangle IYZ, we have $\angle YIZ = \pi - (\alpha - \gamma) - (\frac{\pi}{2} - \alpha) = \frac{\pi}{2} + \gamma$. If Z' is the intersection of line ZI with line XY, then we know that $\angle YIZ' = \pi - (\frac{\pi}{2} + \gamma) = \frac{\pi}{2} - \gamma$. By our assumption that $\beta + \gamma = \frac{\pi}{2}$, we know that this angle is just β. But line l_3 was drawn at angle β from l_2, so line l_2 is in fact our phantom line ZI. Therefore, triangle XYZ is the desired triangle, so this condition is sufficient.

5. Malone has the winning strategy.

Strategy 1

One possible winning strategy is as follows. Malone first picks $a = 1$. If Cooper picks $b = b_0$, then Malone picks $c = b_0^2 + 1$. If Cooper picks $c = c_0$, then Malone picks $b = 1$. Now we prove why this works. Suppose we have a finished game, with polynomial $x^3 + ax^2 + bx + c$. Then this polynomial can be factored as $x^3 + ax^2 + bx + c = (x - r_1)(x - r_2)(x - r_3)$. We know from Viete's relations (or simply expanding the factorization) that

$$\begin{aligned} -a &= r_1 + r_2 + r_3 \\ b &= r_1r_2 + r_2r_3 + r_3r_1 \\ -c &= r_1r_2r_3. \end{aligned}$$

Now note that if we use these equations, we get that $a^2 - 2b = r_1^2 + r_2^2 + r_3^2$ and $b^2 - 2ac = r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_1^2$. If all of the roots are real, then both of these quantities must be greater than or equal to 0 because they are the sums of squares. Therefore, if one of these two quantities is negative, then the polynomial must have a nonreal root. Now if we implement our strategy in the first case, the two quantities are $1 - 2b_0$ and $b_0^2 - (b_0^2 + 1) = -1$ respectively. As the second quantity is negative, the cubic must have a nonreal root. In the second case, the two quantities are $1^2 - 2(1) = -1$ and $1^2 - 2c_0$ respectively. As the first quantity is negative, the cubic must have a nonreal root. Therefore, this strategy gives the cubic a nonreal root, so Malone has a winning strategy.

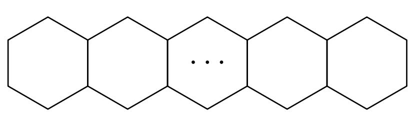
Strategy 2

Another winning strategy is as follows. First, pick $c = 1$. If Cooper picks $b = b_0$, then Malone pick h such that $2h^3 - b_0h - 1 = 0$ (this has a real root because cubics always have at least one real root). Now we write $((x-h)^2+h^2)(x+\frac{1}{2h^2}) = x^3+(\frac{1}{2h^2}-2h)x^2+(2h^2-\frac{1}{h})x+1 = x^3-a_0x^2+b_0x+1$. Therefore, the cubic must have a nonreal root because the roots to the quadratic factor are $x = \frac{2h\pm 2hi}{2}$. If Cooper picks $a = 27$, then Malone picks $b = \frac{1}{27}$. Then the equation factors as $(x^2 + \frac{1}{27})(x + 27)$, which we see has a nonreal root. If Cooper picks $a = a_0$, then we write $(x + \frac{a_0}{3})^3 + (1 - \frac{a_0^3}{27}) = x^3 + a_0x^2 + \frac{a^2}{3}x + 1$. Then this obviously has a nonreal root (find the cube roots of $(1 - \frac{a^3}{27})$). So in any case, Malone has a winning strategy.

Strategy 3 (due to Benjamin Lovelady)

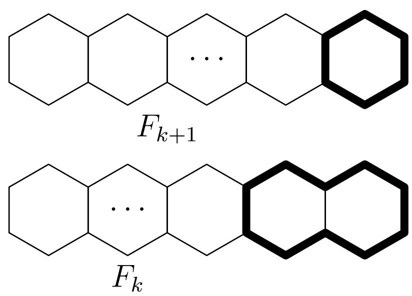
To prove that Malone has the winning strategy, we claim that Malone can always force the cubic to factor as $(x^2 + 1)(x + s)$, where s is a real number. Clearly, this would have a nonreal root, i. This factorization expands to $x^3 + sx^2 + x + s$. Based on the expansion, we see that Malone's first move should be to pick $b = 1$. If Cooper chooses $a = a_0$, then Malone chooses $c = a_0$, yielding the cubic $x^3 + a_0x^2 + x + a_0$. If Cooper chooses $c = c_0$, then Malone chooses $a = c_0$, yielding the cubic $x^3 + c_0x^2 + x + c_0$. It is clear that both cubics are of the desired form, hence each must have a nonreal root. Therefore, Malone has the winning strategy.

6. First, we claim that the number of ways to tile the $1 \times n$ hexagonal strip below is F_n, the n^{th} Fibonacci number, where $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.



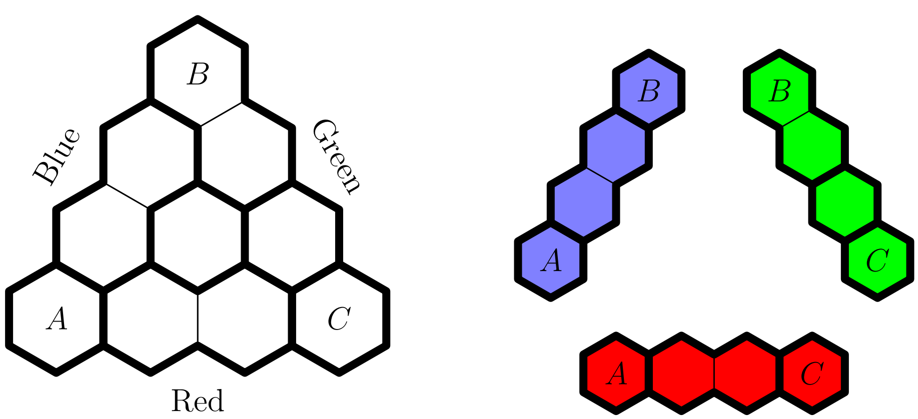
n tiles

This is obviously true for $n = 1$ and $n = 2$ because we have 1 way to tile a 1×1 hexagonal strip and we have two ways to tile a 2×1 hexagonal strip. Now, suppose the statement is true for some k and some $k + 1$. Then a $(k + 2) \times 1$ hexagonal strip can end in either consist of a $(k + 1) \times 1$ strip followed by a 1×1 strip or a $k \times 1$ strip followed by a 2×1 strip. Thus the number of ways to tile a $(k + 2) \times 1$ strip is $F_k + F_{k+1} = F_{k+2}$, so in fact the statement is true for all n, as desired.

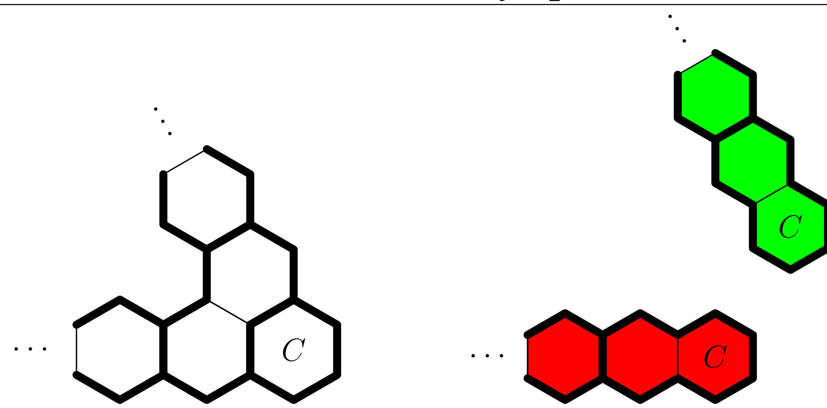


Now we claim that the number of ways to tile the hexagonal border of a triangle where each side is of length n is F_n^3. To demonstrate this, it suffices to demonstrate a one-to-one correspondence between tilings of the hexagonal border and ordered 3-tuples of tilings of a $n \times 1$ hexagonal strip, or in other words, the number of ways to tile three $n \times 1$ hexagonal strips, one colored red, one colored blue, and one colored green.

For a given tiling of the hexagonal border of a triangle, we attempt to map the tiling of the bottom $n \times 1$ strip to a red tiling, the tiling of the left $n \times 1$ strip to a blue tiling, and the tiling of the right $n \times 1$ strip to a green tiling by "pulling" the tiling off. For example, we can map the following tiling of a 4-hexagonal grid as shown below to an RGB $n \times 1$ tiling.



This can be done in the natural way if the bottom, right, and left sides consist of valid $n \times 1$ tilings. But this doesn't always happen. The only case where it can fail, however, is if we have a corner tiled as shown below. In this case, we map it to the shown red and green tilings.



It is easy to see that this will map every tiling of the n-hexagonal grid ($n \geq 3$) to a distinct RGB n-tiling. Furthermore, it is also easy to see that every RGB n-tiling will come from a valid tiling of the n-hexagonal grid. Therefore, this gives us a one to one correspondence between RGB n-tilings and tilings of the n-hexagonal grid. From our work at the beginning, the number of red, green, and blue n-tilings is F_n, so the number of RGB n-tilings is just F_n^3, which is therefore the number of tilings of the n hexagonal grid.