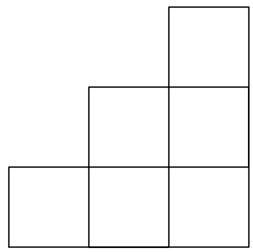
The First Annual Utah Math Olympiad

February 16, 2013

1. Consider the following diagram.



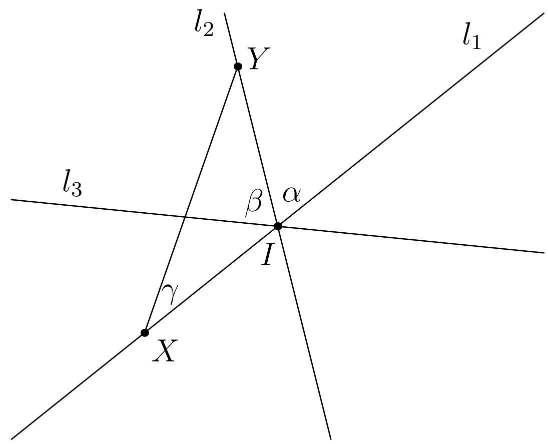
   (a) Show that you can retrace the diagram without lifting up your pencil using exactly nine (possibly overlapping) line segments.

   (b) Show that you cannot retrace the diagram in the same way using eight or fewer segments.

2. Alice and Carl play the following game using a square sheet of paper. On each turn, the player makes a straight cut through the sheet (not necessarily parallel to the sides of the page), creating two new sheets. The sheet with smaller area is discarded (either one if the two are equal), and the player gives the larger sheet to the other player. The first player to receive a sheet of area less than 1 square centimeter from the opposing player loses. If Alice goes first, describe (with proof) the sizes of paper for which she has a winning strategy.

3. Find all x with $1 \leq x \leq 999$ such that the last three digits of x^2 are all equal to the same nonzero digit.

4. Given line l_1 and distinct points I and X on line l_1, draw lines l_2 and l_3 through point I, with angles α and β as marked in the figure. Also, draw line segment XY at an angle of γ from line l_1 such that it intersects line l_2 at Y. Establish necessary and sufficient conditions on α, β, and γ such that a triangle can be drawn with one of its sides as XY with lines l_1, l_2, and l_3 as the angle bisectors of that triangle.



5. Cooper and Malone take turns replacing a, b, and c in the equation below with real numbers.

$$P(x) = x^3 + ax^2 + bx + c$$

Once a coefficient has been replaced, no one can choose to change that coefficient on their turn. The game ends when all three coefficients have been chosen. Malone wins if $P(x)$ has a nonreal root and Cooper wins otherwise. If Malone goes first, find the person who has a winning strategy and describe it with proof.

6. How many ways can one tile the border of a triangular grid of hexagons of length n completely using only 1×1 and 1×2 hexagon tiles? Express your answer in terms of a well-known sequence, and prove that your answer holds true for all positive integers $n \geq 3$ (examples of such grids for $n = 3$, $n = 4$, $n = 5$, and $n = 6$ are shown below).